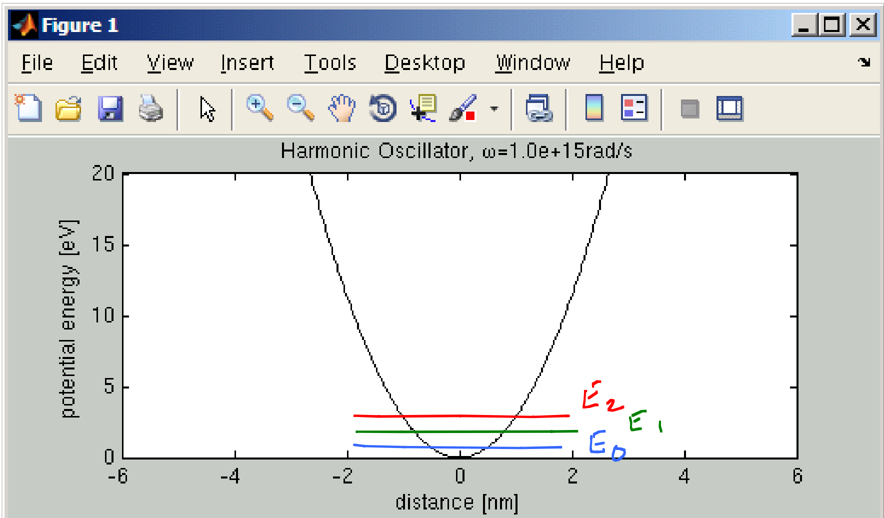
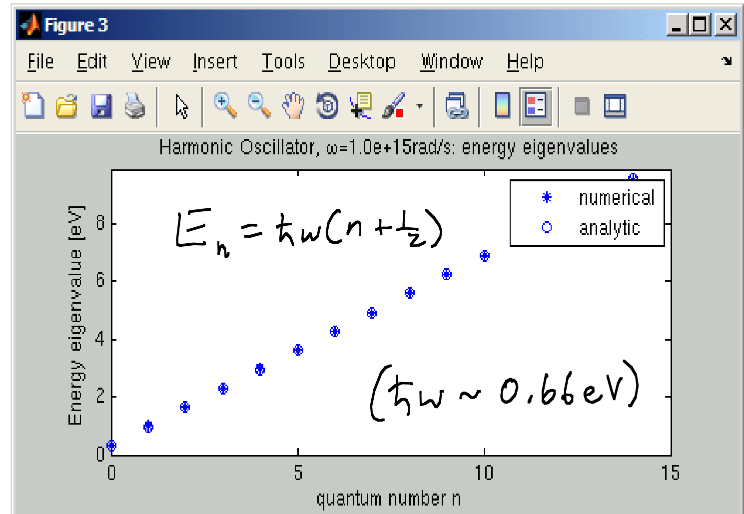
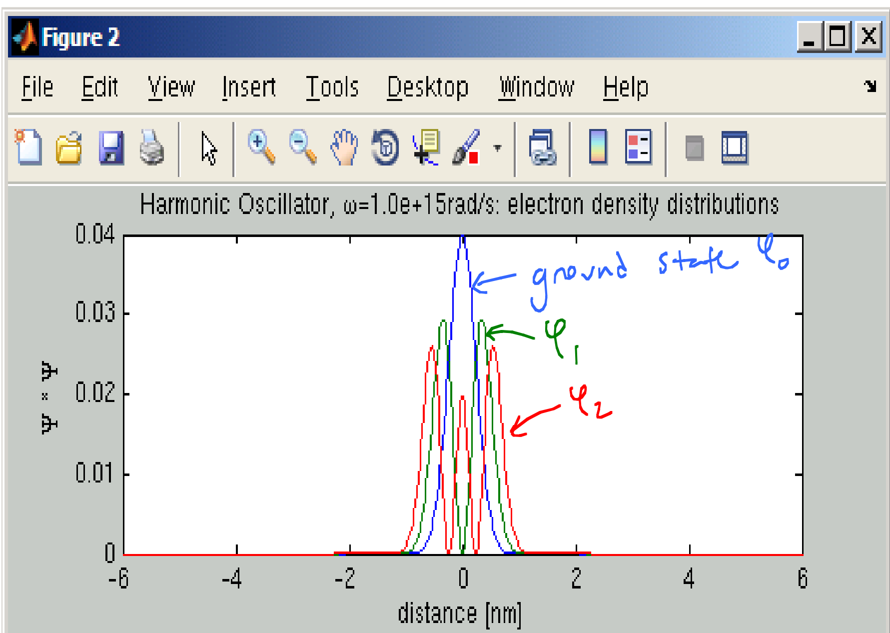
Numerical Solution to Harmonic oscillator







Notice that the probability density $\varphi^*\varphi$ is confined to the bottom of the well, decays into classically forbidden region, and acquires more nodes (where $\varphi^*\varphi = 0$) as n increases.

Ground State properties from Heisenberg uncertainty rel'n

Energy of a wavefunction is determined by the expectation value of Hamiltonian

for Harmonic Oscillator $\langle H \rangle = \frac{\langle p^2 \rangle}{2m} + \frac{1}{2} m \omega^2 \langle x^2 \rangle$.

Since $\langle p \rangle = 0$ for bound state and $\langle x \rangle = 0$ for symmetric potential,

$$\Delta x = \sqrt{\langle x^2 \rangle - \cancel{\langle x \rangle^2}^{\,0}} \longrightarrow \langle x^2 \rangle = \Delta x^2 \quad \text{and} \quad \Delta p = \sqrt{\langle p^2 \rangle - \cancel{\langle p \rangle^2}^{\,0}} \longrightarrow \langle p^2 \rangle = \Delta p^2$$

So $\langle H \rangle = \frac{\Delta p^2}{2m} + \frac{1}{2} m \omega^2 \Delta x^2$ but we don't know either Δx or Δp.

However, we know the two are connected by the Heisenberg rel'n:

$\Delta x \Delta p \geq \frac{\hbar}{2}$. Using $\Delta p \geq \frac{\hbar}{2\Delta x}$, we have

$$\langle H \rangle \leq \frac{\hbar^2}{8m\Delta x^2} + \frac{1}{2} m \omega^2 \Delta x^2$$

The <u>ground-state</u> energy must be lower than the <u>minimum</u> value of the RHS.

Minimization

$$\frac{d}{d\Delta x}\left(\frac{\hbar^2}{8m\Delta x^2} + \frac{1}{2}m\omega^2\Delta x^2\right) = -\frac{\hbar^2}{4m\Delta x^3} + m\omega^2\Delta x = 0$$ which has sol'n $\Delta x = \sqrt{\frac{\hbar}{2m\omega}}$

Let's compare to the exact solution $\psi_0 \propto e^{-\frac{m\omega x^2}{2\hbar}}$:

Same!

probability density $\psi_0^* \psi_0 \propto e^{-\frac{m\omega x^2}{\hbar}} = e^{-\frac{x^2}{2\Delta x^2}}$ where $\Delta x = \sqrt{\frac{\hbar}{2m\omega}}$

We can also calculate the upper bound on energy of the ground state:

$$\langle H \rangle \leq \frac{\hbar^2}{8m\Delta x^2} + \frac{1}{2}m\omega^2\Delta x^2 = \frac{\hbar^2}{8m\frac{\hbar}{2m\omega}} + \frac{1}{2}m\omega^2\frac{\hbar}{2m\omega} = \frac{\hbar\omega}{4} + \frac{\hbar\omega}{4} = \frac{\hbar\omega}{2}$$

Same as analytic value for E_0!

In this case, the inequality is an <u>equality</u> because the gaussian ground state minimizes the product $\Delta x \Delta p$!

In general, of course, ground state wavefunctions are not gaussian and this procedure only gives an upper bound.