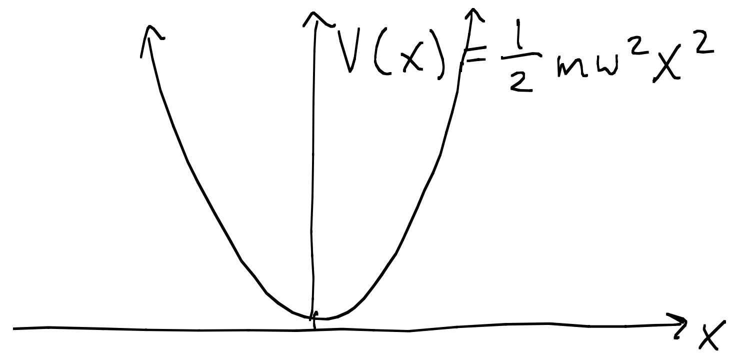
Harmonic Oscillator



$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 = \frac{1}{2m}\left(p^2 + (m\omega x)^2\right)$$

Try to factor H:

$$H \overset{?}{=} \frac{1}{2m}(ip + m\omega x)(-ip + m\omega x) = \frac{1}{2m}\left(p^2 + (m\omega x)^2 + ipm\omega x + m\omega x(-ip)\right)$$

$$= \frac{1}{2m}\left(p^2 + (m\omega x)^2 + im\omega\,\underbrace{(px - xp)}_{[p,x]}\right)$$

Note That $[p,x]\varphi = \frac{\hbar}{i}\left(\frac{d}{dx}(x\varphi) - x\frac{d}{dx}\varphi\right) = \frac{\hbar}{i}\left(\varphi + x\frac{d\varphi}{dx} - x\frac{d\varphi}{dx}\right) = \frac{\hbar}{i}\varphi$

So $[p,x] = \frac{\hbar}{i} = -i\hbar$ and

$$H = \frac{1}{2m}\left[(ip + m\omega x)(-ip + m\omega x) - m\hbar\omega\right]$$

$$H = \hbar\omega\left[\underbrace{\frac{1}{\sqrt{2m\hbar\omega}}(ip + m\omega x)}_{a_-}\,\underbrace{\frac{1}{\sqrt{2m\hbar\omega}}(-ip + m\omega x)}_{a_+} - \frac{1}{2}\right] = \hbar\omega\left(a_- a_+ - \frac{1}{2}\right)$$

"Ladder operators"

$$a_- a_+ = \frac{H}{\hbar\omega} + \frac{1}{2}$$

$$a_+ a_- = \frac{H}{\hbar\omega} - \frac{1}{2} \qquad (\text{since } [p,x] = -[x,p])$$

$$[a_-, a_+] = 1$$

If ψ is an eigenfunction of H, $H\psi = E\psi$. Then, what is $a_+\psi = ?$

$$H(a_+\psi) = \hbar\omega\left(a_+a_- + \tfrac{1}{2}\right)(a_+\psi) = \hbar\omega\left(a_+a_-a_+ + \tfrac{1}{2}a_+\right)\psi$$

$$= a_+\hbar\omega\left(a_-a_+ + \tfrac{1}{2}\right)\psi = a_+\hbar\omega\left(\frac{H}{\hbar\omega} + \frac{1}{2} + \frac{1}{2}\right)\psi$$

$$= a_+(H + \hbar\omega)\psi = (E + \hbar\omega)(a_+\psi)$$

So $a_+\psi$ is also an eigenfunction of H, with eigenvalue greater by $\hbar\omega$! We therefore call a_+ the "raising" operator.

Likewise,

$$H(a_-\varphi) = \hbar\omega\left(a_-a_+ - \tfrac{1}{2}\right)(a_-\varphi) = \hbar\omega\left(a_-a_+a_- - \tfrac{1}{2}a_-\right)\varphi$$

$$= a_-\hbar\omega\left(a_+a_- - \tfrac{1}{2}\right)\varphi = a_-\hbar\omega\left(\frac{H}{\hbar\omega} - \frac{1}{2} - \frac{1}{2}\right)\varphi$$

$$= a_-(H - \hbar\omega)\varphi = a_-(E - \hbar\omega)\varphi = (E - \hbar\omega)(a_-\varphi)$$

So $a_-\varphi$ is also an eigenfunction of H, with eigenvalue reduced by $\hbar\omega$! We therefore call a_- the "lowering" operator.

Eventually, $a_- \varphi_0 = 0$

$$\frac{1}{\sqrt{2m\hbar\omega}}(ip + m\omega x)\varphi_0 = 0$$

$$\left(\hbar\frac{d}{dx} + m\omega x\right)\varphi_0 = 0$$

$$\frac{d}{dx}\varphi_0 = -\frac{m\omega x}{\hbar}\varphi_0$$

This is a 1st-order differential eqn we can use to find φ_0:

$$\int\frac{d\varphi_0}{\varphi_0} = -\int\frac{m\omega x}{\hbar}dx$$

$$\ln\varphi_0 = -\frac{m\omega}{\hbar}\frac{x^2}{2} + C$$

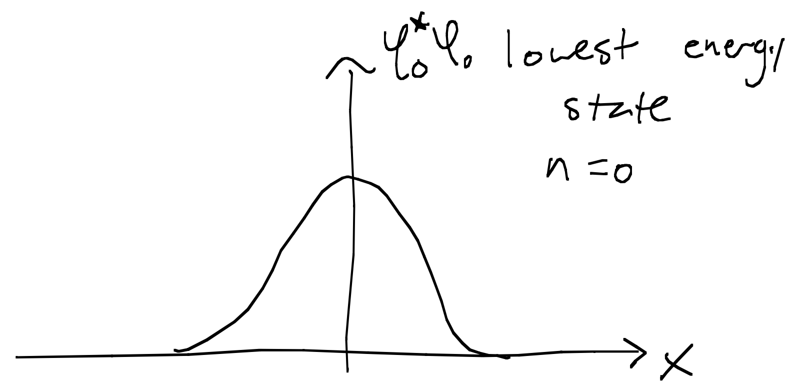
$$\varphi_0 = Ae^{-\frac{m\omega x^2}{2\hbar}}$$ gaussian "ground state"

Normalization:

$$\int_{-\infty}^{\infty}\varphi_0^*\varphi_0\,dx = \int_{-\infty}^{\infty}A^2e^{-\frac{m\omega}{\hbar}x^2}dx$$

$$= A^2\sqrt{\frac{\pi\hbar}{m\omega}} = 1$$

$$\Rightarrow A = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$$



Generate ψ_1

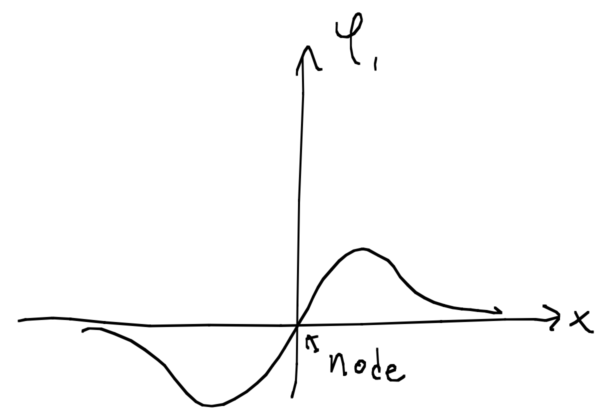
$$\psi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}}$$

$$a_+ = \frac{1}{\sqrt{2m\hbar\omega}}\left(-ip + m\omega x\right)$$

$$\psi_1 = a_+\psi_0 = \frac{1}{\sqrt{2m\hbar\omega}}\left(-\cancel{i}\frac{\hbar}{\cancel{i}}\frac{d}{dx} + m\omega x\right)\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}}$$

$$= \left(\frac{1}{4m\omega\pi\hbar^3}\right)^{1/4}\left(\cancel{\hbar}\frac{m\omega}{\cancel{\hbar}}x + m\omega x\right) e^{-\frac{m\omega x^2}{2\hbar}}$$

$$= \left[\left(\frac{m\omega}{\hbar}\right)^3 \frac{4}{\pi}\right]^{1/4} x\, e^{-\frac{m\omega x^2}{2\hbar}}$$



Note: $\int_{-\infty}^{\infty} \psi_1^* \psi_1 \, dx = \frac{2}{\sqrt{\pi}}\left(\frac{m\omega}{\hbar}\right)^{3/2}\int_{-\infty}^{\infty} x^2 e^{-\frac{m\omega x^2}{\hbar}} dx = \frac{2}{\sqrt{\pi}}\, \xi^{3/2}\int_{-\infty}^{\infty} x^2 e^{-\xi x^2} dx = \frac{2}{\sqrt{\pi}}\, \xi^{3/2}\left[\frac{\sqrt{\pi}}{2\xi^{3/2}}\right] = 1$

$\left(\xi = \frac{m\omega}{\hbar}\right)$

However, higher ψ_n are not automatically normalized...

Spectrum

Since $a_- \varphi_0 = 0$, $H\varphi_0 = \hbar\omega\left(a_+ a_- + \frac{1}{2}\right)\varphi_0 = \frac{\hbar\omega}{2}\varphi_0$.

Higher eigenvalues are spaced by $\hbar\omega$, so we have

$$E_n = \left(n + \tfrac{1}{2}\right)\hbar\omega, \quad n = 0, 1, 2, \ldots$$

(This is equivalent to recognizing that the eigenvalues of $a_+ a_-$ are $n = 0, 1, 2 \ldots$)