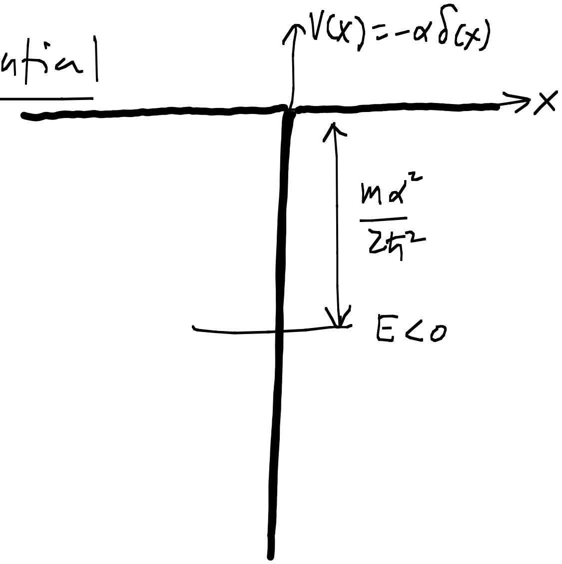
For bound state of attractive delta fn potential



$$-\frac{\hbar^2}{2m}\varphi'' = E\varphi \qquad (x \neq 0)$$

$$x<0: \quad \varphi_-(x) = Ae^{\kappa x} + \cancel{B}^{\,0} e^{-\kappa x}, \qquad \kappa = \sqrt{\frac{-2mE}{\hbar^2}}$$

$$x>0: \quad \varphi_+(x) = \cancel{A'}^{\,0} e^{\kappa x} + B' e^{-\kappa x}$$

B.C.'s: $\varphi \to 0$ as $x \to \pm\infty$ so that it is normalizable $(A' = B = 0)$

φ is continuous across $x=0$ $\quad (A = B')$

$$\varphi_+'(0) - \varphi_-'(0) = -\frac{2m\alpha}{\hbar^2}\varphi(0)$$

$$\varphi_+'(0) - \varphi_-'(0) = -\kappa B' - \kappa A = -2\kappa A = -\frac{2m\alpha}{\hbar^2}A \longrightarrow \kappa = \frac{m\alpha}{\hbar^2}$$

$$E = -\frac{\hbar^2\kappa^2}{2m} = -\frac{m\alpha^2}{2\hbar^2}$$

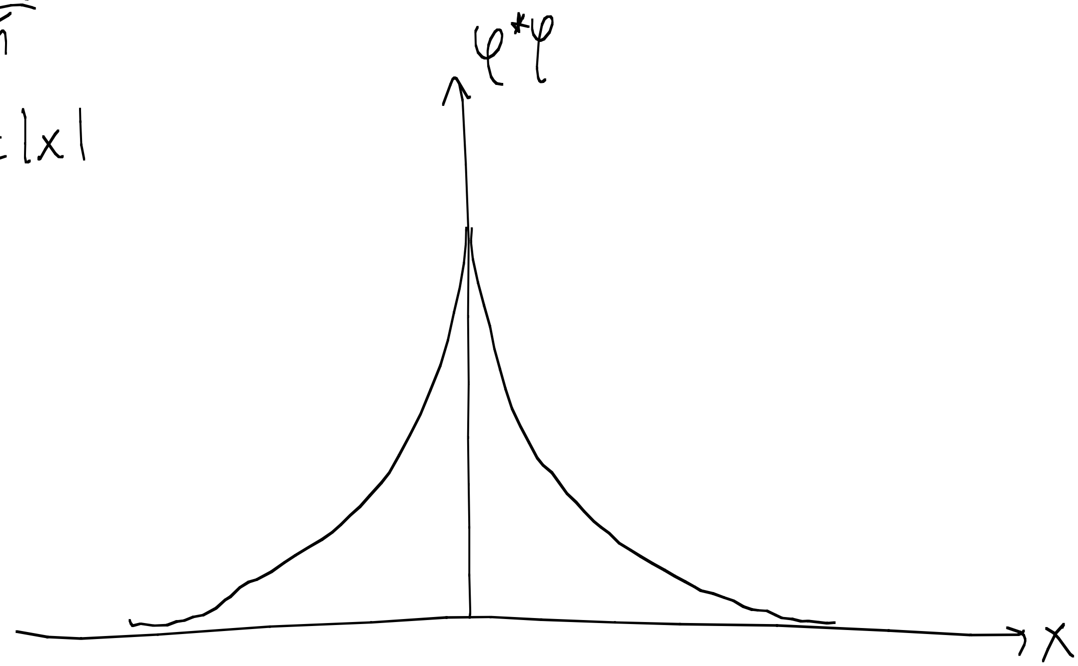
Normalization of Delta-function bound state

$$\psi_-(x) = Ae^{\kappa x}, \qquad \psi_+(x) = Ae^{-\kappa x}$$

$$\int_{-\infty}^{\infty} \psi^* \psi \, dx = \int_{-\infty}^{0} \psi_-^* \psi_- \, dx + \int_0^{+\infty} \psi_+^* \psi_+ \, dx = 2\int_0^{\infty} A^2 e^{-2\kappa x} dx = -\frac{2A^2}{2\kappa} e^{-2\kappa x}\Big|_0^{\infty} = \frac{A^2}{\kappa} = 1$$

so $A = \sqrt{\kappa} = \frac{\sqrt{m\alpha}}{\hbar}$

$$\psi(x) = \frac{\sqrt{m\alpha}}{\hbar} e^{-\frac{m\alpha}{\hbar^2}|x|}$$



Numerical Result

Since $-\alpha\delta(x)$ is the analytic limit of a sequence of narrower, deeper wells with constant $\alpha = \int_0^{\text{width}} V(x)\,dx$, we can use our matrix eigenvalue scheme to numerically approximate the solution: