Direct Calculation of Δx (free particle wavefunction)

$$\Psi(x,t) = \left(\frac{2a}{\pi}\right)^{1/4} \frac{1}{\sqrt{1+4iDat}}\, e^{-\frac{ax^2}{1+4iDat}} \qquad \left(D \equiv \frac{\hbar}{2m}\right)$$

$$\text{"}\Delta x\text{"} = \sigma_x = \sqrt{\langle x^2\rangle - \cancelto{0}{\langle x\rangle^2}}$$

$$\langle x^2\rangle = \int_{-\infty}^{\infty} \Psi^* x^2 \Psi\, dx = \sqrt{\frac{2a}{\pi(1+16D^2a^2t^2)}} \int_{-\infty}^{\infty} x^2 e^{-\frac{2ax^2}{1+16D^2a^2t^2}}\, dx$$

$$= \sqrt{\frac{\xi}{\pi}} \int_{-\infty}^{\infty} x^2 e^{-\xi x^2}\, dx \qquad \xi = \frac{2a}{1+16D^2a^2t^2}$$

(differentiate under integral) $$= \sqrt{\frac{\xi}{\pi}} \int_{-\infty}^{\infty} \left(-\frac{d}{d\xi}\right) e^{-\xi x^2}\, dx = -\sqrt{\frac{\xi}{\pi}} \frac{d}{d\xi}\left[\int_{-\infty}^{\infty} e^{-\xi x^2}\, dx\right]$$

$$= -\sqrt{\frac{\xi}{\pi}} \frac{d}{d\xi} \sqrt{\frac{\pi}{\xi}} = \sqrt{\frac{\xi}{\pi}}\, \frac{1}{2} \frac{\sqrt{\pi}}{\xi^{3/2}} = \frac{1}{2\xi} = \frac{1+16D^2a^2t^2}{4a}$$

So $$\text{"}\Delta x\text{"} = \sigma_x = \sqrt{\langle x^2\rangle} = \sqrt{\frac{1+16D^2a^2t^2}{4a}}$$

Verifying the Heisenberg Uncertainty principle

$$\text{"}\Delta p\text{"} = \sigma_p = \sqrt{\langle p^2\rangle - \cancel{\langle p\rangle}^2} \quad \xrightarrow{} 0 \text{ by symmetry}$$

$$\Psi(x,t) = \left(\frac{2a}{\pi}\right)^{1/4} \frac{1}{\sqrt{1+4iDat}}\, e^{-\frac{ax^2}{1+4iDat}}$$

$$\langle p^2\rangle = \int_{-\infty}^{\infty} \Psi^* \left(\frac{\hbar}{i}\frac{d}{dx}\right)^2 \Psi\, dx = -\hbar^2 \int_{-\infty}^{\infty} \Psi^* \frac{d}{dx}\left(-\frac{2ax}{1+4iDat}\Psi\right) dx$$

$$= \frac{2a\hbar^2}{1+4iDat} \int_{-\infty}^{\infty} \Psi^* \left(\Psi + x\left(-\frac{2ax}{1+4iDat}\Psi\right)\right) dx = \frac{2a\hbar^2}{1+4iDat}\left[\underbrace{\int_{-\infty}^{\infty} \Psi^*\Psi\, dx}_{1} - \frac{2a}{1+4iDat} \underbrace{\int_{-\infty}^{\infty} \Psi^* x^2 \Psi\, dx}_{\langle x^2\rangle}\right]$$

$$= \frac{2a\hbar^2}{1+4iDat}\left[1 - \frac{2a}{1+\cancel{4iD}at}\,\frac{\cancel{(1+4iDat)}(1-4iDat)}{4a}\right] = \frac{\cancel{2}a\hbar^2}{\cancel{1+4iDat}}\,\frac{1}{\cancel{2}}\left(\cancel{1+4iDat}\right) = a\hbar^2$$

So $\text{"}\Delta p\text{"} = \sigma_p = \sqrt{\langle p^2\rangle} = \hbar\sqrt{a}$ time independent!

Spatial translation invariance → Momentum is conserved

$$\text{"}\Delta x\text{"}\,\text{"}\Delta p\text{"} = \sigma_x \sigma_p = \sqrt{\frac{1+16D^2a^2t^2}{4a}}\,\hbar\sqrt{a} = \frac{\hbar}{2}\cdot\left(1+16D^2a^2t^2\right)^{1/2} \geq \frac{\hbar}{2} \quad \checkmark$$

General Case $V(x) \neq 0$

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) + V(x)\Psi(x,t) = i\hbar\frac{\partial}{\partial t}\Psi(x,t)$$

Separation of variables: $\Psi(x,t) = \varphi(x)\phi(t)$

$$\frac{-\frac{\hbar^2}{2m}\varphi''\phi + V\varphi\phi}{\varphi\phi} = \frac{i\hbar\varphi\dot{\phi}}{\varphi\phi}$$

$$\frac{-\frac{\hbar^2}{2m}\varphi'' + V\varphi}{\varphi} = \frac{i\hbar\dot{\phi}}{\phi} = E$$

only if $V(x,t) \rightarrow V(x)$

time invariance $\longrightarrow$ energy conservation

For x:

$$-\frac{\hbar^2}{2m}\varphi'' + V(x)\varphi = E\varphi$$

"Time-independent Schrödinger Eqn"

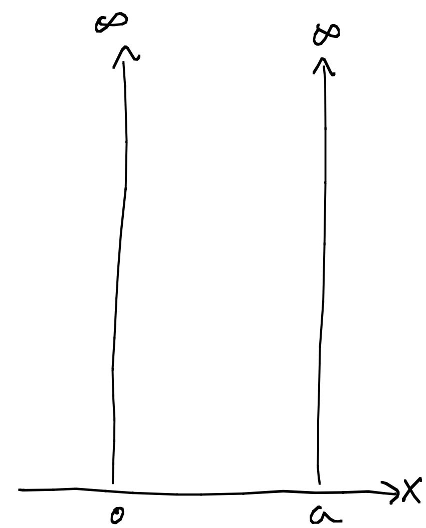
For t:

$$\dot{\phi} = -i\frac{E}{\hbar}\phi \longrightarrow \phi(t) = Ce^{-i\frac{E}{\hbar}t}$$

In "stationary" potential, time dependence of $\Psi(x,t)$ is just an overall phase!

Infinite Square well

$$V(x) = \begin{cases} 0 & 0<x<a \\ \infty & \text{otherwise} \end{cases}$$



$$\psi'' = -\frac{2m(E-V)}{\hbar^2}\psi = -k^2\psi$$

Solutions: $\psi(x) = A e^{ikx} + B e^{-ikx}$, $k \equiv \sqrt{\frac{2m(E-V)}{\hbar^2}}$

In regions $x<0$ and $x>a$, $k = \pm i$"∞" so $\psi \to \infty$ unless $A, B = 0$

This means we have B.C.'s: $\psi(0) = 0$ and $\psi(a) = 0$

$\psi(x) = A' \sin kx + B' \cos kx \qquad (0<x<a)$

Impose $\psi(0) = B' = 0$

$\psi(a) = A' \sin ka = 0 \longrightarrow k_n = \frac{n\pi}{a} \qquad n = 1, 2, 3, \ldots$ "quantum number"

From k_n, we determine the Energy Eigenvalue $\longrightarrow$ $E = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2 n^2 \pi^2}{2ma^2}$

Normalization

$$\int_{-\infty}^{\infty} \psi^* \psi \, dx = 1$$

$$\int_0^a A'^2 \sin^2 kx \, dx = A'^2 \int_0^a \frac{1 - \cos 2kx}{2} dx = A'^2 \left[\frac{x}{2} - \frac{\sin 2kx}{4k} \right] \Bigg|_0^a = A'^2 \frac{a}{2} = 1$$

So $\quad A' = \sqrt{\frac{2}{a}}$

eigenfunctions and associated eigenvalues are

$$\varphi_n(x) = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a}, \quad \Psi_n(x,t) = \begin{cases} \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a} \, e^{-\frac{iE_n}{\hbar} t}, & 0 < x < a, \quad E_n = \frac{\hbar^2 \pi^2 n^2}{2ma^2} \\ 0 & \text{otherwise} \end{cases}$$

Remember: Any wavefunction $\Psi(x,t)$ can be decomposed into a linear superposition of eigenfunctions $\Psi_n(x,t)$