Expectation values of observables

In QM, every observable has a corresponding operator

For any observable Q, the possible outcome of measurement are the eigenvalues of the corresponding operator $\hat{Q}$

$$\hat{Q}\chi_i = q_i\chi_i$$ (χ_i are eigenfunctions/eigenvectors and q_i are corresponding eigenvalues)

But, in general, Ψ is not an eigenfunction. But, $\Psi = \sum_i b_i\chi_i$ so

$$\hat{Q}\Psi = \sum_i b_i q_i \chi_i$$ Now take inner product w wavefunction

$$\int_{-\infty}^{\infty}\Psi^*\hat{Q}\Psi dx = \int_{-\infty}^{\infty}\Psi^*\sum_i b_i q_i \chi_i dx = \int_{-\infty}^{\infty}\sum_j b_j^*\chi_j^*\sum_i b_i q_i\chi_i dx$$

$$= \sum_i\sum_j b_j^* b_i q_i \int_{-\infty}^{\infty}\chi_j^*\chi_i dx \xrightarrow{\text{orthonormality}} = \sum_i\sum_j b_j^* b_i q_i \delta_{ij}$$

$$= \sum_i |b_i|^2 q_i = \langle Q\rangle$$ "expectation value" is a weighted sum of all possibilities!

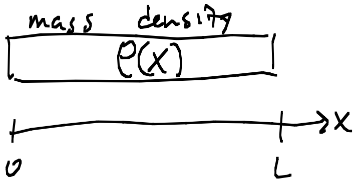
So $$\langle Q\rangle = \int_{-\infty}^{\infty}\Psi^* Q\Psi dx$$

Examples Associated with mechanical motion

position $\quad \langle x \rangle = \int_{-\infty}^{\infty} \psi^* x \psi \, dx = \int_{-\infty}^{\infty} x \, \psi^* \psi \, dx$

Since $\psi^*\psi$ is probability density, This is the same as classical distribution theory!

e.g.



center of mass $\quad \langle x \rangle = \dfrac{\int_0^L x \rho(x)\, dx}{\int_0^L \rho(x)\, dx}$ $\quad$ (Note That $\int_{-\infty}^{\infty} \psi^*\psi \, dx = 1$)

velocity $\quad \langle v \rangle = \dfrac{d\langle x \rangle}{dt} = \int_{-\infty}^{\infty} x \dfrac{d}{dt} \underbrace{(\psi^*\psi)}_{\text{probability density}} dx \xrightarrow{\text{Continuity}} \int_{-\infty}^{\infty} x \left(- \dfrac{d \overbrace{J}^{\text{probability current}}}{dx} \right) dx$

$$= \cancelto{0}{-J(x)\, x \Big|_{-\infty}^{\infty}} + \int_{-\infty}^{\infty} J \, dx \quad \text{where } J = \frac{\hbar}{2mi} \left(\psi^* \frac{d}{dx}\psi - \frac{d}{dx}\psi^* \cdot \psi \right)$$

Since ψ must be normalized, $\psi(x = \pm\infty) \to 0$ $\quad$ so $\quad J(x = \pm\infty) = 0$

$$= \frac{\hbar}{2mi} \left[\int_{-\infty}^{\infty} \psi^* \frac{d}{dx} \psi \, dx - \int_{-\infty}^{\infty} \frac{d}{dx} \psi^* \cdot \psi \, dx \right]$$

Integration by parts on second term yields:

$$-\int_{-\infty}^{\infty} \frac{d}{dx}\Psi^* \cdot \Psi \, dx = -\cancelto{0}{\Psi^*\Psi\Big|_{-\infty}^{\infty}} + \int_{-\infty}^{\infty} \Psi^* \frac{d}{dx}\Psi \, dx \quad \Rightarrow \text{The first term!}$$

So

$$\langle v \rangle = \frac{\hbar}{mi}\int_{-\infty}^{\infty} \Psi^* \frac{d}{dx}\Psi \, dx$$

Therefore

$$\langle p \rangle = m\langle v \rangle = \int_{-\infty}^{\infty} \Psi^* \frac{\hbar}{i}\frac{d}{dx}\Psi \, dx$$

As expected, since $\hat{p} = \frac{\hbar}{i}\frac{d}{dx}$!

We could have just used $\int_{-\infty}^{\infty} \Psi^* Q \Psi \, dx$!

Uncertainty from expectation values

The standard deviation of observable Q is given by

$$\sigma_Q = (\text{variance})^{1/2} = \left[\int_{-\infty}^{\infty} \Psi^* \left(Q - \langle Q \rangle\right)^2 \Psi \, dx \right]^{1/2}$$

$$= \left[\int_{-\infty}^{\infty} \Psi^* \left(Q^2 - 2Q\langle Q \rangle + \langle Q \rangle^2\right) \Psi \, dx \right]^{1/2}$$

$$= \left[\int_{-\infty}^{\infty} \Psi^* Q^2 \Psi \, dx - 2\langle Q \rangle \int_{-\infty}^{\infty} \Psi^* Q \Psi \, dx + \langle Q \rangle^2 \int \Psi^* \Psi \, dx \right]^{1/2}$$

$$= \left(\langle Q^2 \rangle - 2\langle Q \rangle^2 + \langle Q \rangle^2 \right)^{1/2} = \left(\langle Q^2 \rangle - \langle Q \rangle^2 \right)^{1/2}$$

For gaussian state above,

$$\sigma_x = \sqrt{\langle x^2 \rangle - \cancel{\langle x \rangle^2}^{\,0}}$$

$$\sigma_p = \sqrt{\langle p^2 \rangle - \cancel{\langle p \rangle^2}^{\,0}}$$

by symmetry