Problem 2.34

(a)

$$\psi = \begin{Bmatrix} Ae^{ikx} + Be^{-ikx} & (x < 0) \\ Fe^{-\kappa x} & (x > 0) \end{Bmatrix} \text{ where } k = \frac{\sqrt{2mE}}{\hbar};\ \kappa = \frac{\sqrt{2m(V_0 - E)}}{\hbar}.$$

(1) Continuity of ψ : $A + B = F$.

(2) Continuity of ψ' : $ik(A - B) = -\kappa F$.

$$\Rightarrow A + B = -\frac{ik}{\kappa}(A - B) \Rightarrow A\left(1 + \frac{ik}{\kappa}\right) = -B\left(1 - \frac{ik}{\kappa}\right).$$

$$R = \left|\frac{B}{A}\right|^2 = \frac{|(1 + ik/\kappa)|^2}{|(1 - ik/\kappa)|^2} = \frac{1 + (k/\kappa)^2}{1 + (k/\kappa)^2} = \boxed{1.}$$

Although the wave function penetrates into the barrier, it is eventually all reflected.

(b)

$$\psi = \begin{Bmatrix} Ae^{ikx} + Be^{-ikx} & (x < 0) \\ Fe^{ilx} & (x > 0) \end{Bmatrix} \text{ where } k = \frac{\sqrt{2mE}}{\hbar};\ l = \frac{\sqrt{2m(E - V_0)}}{\hbar}.$$

(1) Continuity of ψ : $A + B = F$.

(2) Continuity of ψ' : $ik(A - B) = ilF$.

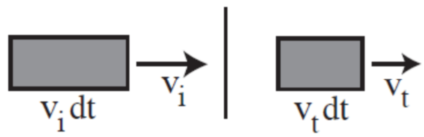
$$\Rightarrow A + B = \frac{k}{l}(A - B);\ A\left(1 - \frac{k}{l}\right) = -B\left(1 + \frac{k}{l}\right).$$

$$R = \left|\frac{B}{A}\right|^2 = \frac{(1 - k/l)^2}{(1 + k/l)^2} = \frac{(k - l)^2}{(k + l)^2} = \frac{(k - l)^4}{(k^2 - l^2)^2}.$$

Now $k^2 - l^2 = \frac{2m}{\hbar^2}(E - E + V_0) = \left(\frac{2m}{\hbar^2}\right) V_0;\ k - l = \frac{\sqrt{2m}}{\hbar}[\sqrt{E} - \sqrt{E - V_0}],$ so

$$\boxed{R = \frac{(\sqrt{E} - \sqrt{E - V_0})^4}{V_0^2}.}$$

(c)



From the diagram, $T = P_t/P_i = |F|^2 v_t/|A|^2 v_i$, where P_i is the probability of finding the incident particle in the box corresponding to the time interval dt, and P_t is the probability of finding the transmitted particle in the associated box to the *right* of the barrier.

But $\frac{v_t}{v_i} = \frac{\sqrt{E-V_0}}{\sqrt{E}}$ (from Eq. 2.98). So $T = \sqrt{\frac{E-V_0}{E}}\left|\frac{F}{A}\right|^2$. Alternatively, from Problem 2.19:

$$J_i = \frac{\hbar k}{m}|A|^2; \quad J_t = \frac{\hbar l}{m}|F|^2; \quad T = \frac{J_t}{J_i} = \left|\frac{F}{A}\right|^2 \frac{l}{k} = \left|\frac{F}{A}\right|^2 \sqrt{\frac{E-V_0}{E}}.$$

For $E < V_0$, of course, $\boxed{T = 0.}$

(d)

$$\text{For } E > V_0,\ F = A + B = A + A\frac{(\frac{k}{l}-1)}{(\frac{k}{l}+1)} = A\frac{2k/l}{(\frac{k}{l}+1)} = \frac{2k}{k+l}A.$$

$$T = \left|\frac{F}{A}\right|^2 \frac{l}{k} = \left(\frac{2k}{k+l}\right)^2 \frac{l}{k} = \frac{4kl}{(k+l)^2} = \frac{4kl(k-l)^2}{(k^2-l^2)^2} = \boxed{\frac{4\sqrt{E}\sqrt{E-V_0}(\sqrt{E}-\sqrt{E-V_0})^2}{V_0^2}.}$$

$$T + R = \frac{4kl}{(k+l)^2} + \frac{(k-l)^2}{(k+l)^2} = \frac{4kl + k^2 - 2kl + l^2}{(k+l)^2} = \frac{k^2 + 2kl + l^2}{(k+l)^2} = \frac{(k+l)^2}{(k+l)^2} = 1. \checkmark$$

Problem 2.35

(a)

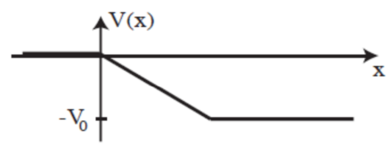
$$\psi(x) = \begin{Bmatrix} Ae^{ikx} + Be^{-ikx} & (x<0) \\ Fe^{ilx} & (x>0) \end{Bmatrix} \text{ where } k \equiv \frac{\sqrt{2mE}}{\hbar},\ l \equiv \frac{\sqrt{2m(E+V_0)}}{\hbar}.$$

$$\left.\begin{array}{l} \text{Continuity of } \psi \ \Rightarrow A + B = F \\ \text{Continuity of } \psi' \Rightarrow ik(A-B) = ilF \end{array}\right\} \Longrightarrow$$

$$A + B = \frac{k}{l}(A-B); \quad A\left(1 - \frac{k}{l}\right) = -B\left(1 + \frac{k}{l}\right); \quad \frac{B}{A} = -\left(\frac{1-k/l}{1+k/l}\right).$$

$$R = \left|\frac{B}{A}\right|^2 = \left(\frac{l-k}{l+k}\right)^2 = \left(\frac{\sqrt{E+V_0}-\sqrt{E}}{\sqrt{E+V_0}+\sqrt{E}}\right)^2$$
$$= \left(\frac{\sqrt{1+V_0/E}-1}{\sqrt{1+V_0/E}+1}\right)^2 = \left(\frac{\sqrt{1+3}-1}{\sqrt{1+3}+1}\right)^2 = \left(\frac{2-1}{2+1}\right)^2 = \boxed{\frac{1}{9}}.$$

(b) The cliff is *two*-dimensional, and even if we pretend the car drops straight down, the potential *as a function of distance along the* (crooked, but now one-dimensional) *path* is $-mgx$ (with x the vertical coordinate), as shown.



(c) Here $V_0/E = 12/4 = 3$, the same as in part (a), so $R = 1/9$, and hence $T = \boxed{8/9 = 0.8889.}$

Problem 2.42

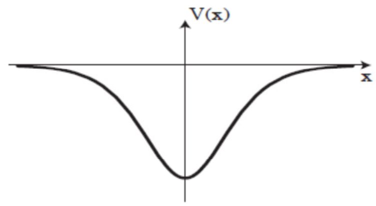
Everything in Section 2.3.2 still applies, except that there is an additional boundary condition: $\psi(0) = 0$. This eliminates all the *even* solutions ($n = 0, 2, 4, \ldots$), leaving only the odd solutions. So

$$\boxed{E_n = \left(n + \frac{1}{2}\right)\hbar\omega, \; n = 1, 3, 5, \ldots .}$$

Problem 2.51

(a) Figure at top of next page.

(b) $\dfrac{d\psi_0}{dx} = -Aa\,\mathrm{sech}(ax)\tanh(ax)$; $\quad \dfrac{d^2\psi_0}{dx^2} = -Aa^2\left[-\mathrm{sech}(ax)\tanh^2(ax) + \mathrm{sech}(ax)\,\mathrm{sech}^2(ax)\right].$

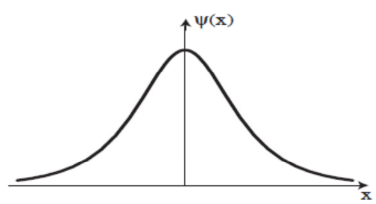


$$
\begin{aligned}
H\psi_0 &= -\frac{\hbar^2}{2m}\frac{d^2\psi_0}{dx^2} - \frac{\hbar^2 a^2}{m}\operatorname{sech}^2(ax)\psi_0 \\
&= \frac{\hbar^2}{2m}Aa^2\left[-\operatorname{sech}(ax)\tanh^2(ax) + \operatorname{sech}^3(ax)\right] - \frac{\hbar^2 a^2}{m}A\operatorname{sech}^3(ax) \\
&= \frac{\hbar^2 a^2 A}{2m}\left[-\operatorname{sech}(ax)\tanh^2(ax) + \operatorname{sech}^3(ax) - 2\operatorname{sech}^3(ax)\right] \\
&= -\frac{\hbar^2 a^2}{2m}A\operatorname{sech}(ax)\left[\tanh^2(ax) + \operatorname{sech}^2(ax)\right].
\end{aligned}
$$

But $(\tanh^2\theta + \operatorname{sech}^2\theta) = \dfrac{\sinh^2\theta}{\cosh^2\theta} + \dfrac{1}{\cosh^2\theta} = \dfrac{\sinh^2\theta + 1}{\cosh^2\theta} = 1$, so

$$= -\frac{\hbar^2 a^2}{2m}\psi_0. \quad \text{QED} \quad \text{Evidently } \boxed{E = -\frac{\hbar^2 a^2}{2m}.}$$

$$1 = |A|^2 \int_{-\infty}^{\infty} \operatorname{sech}^2(ax)dx = |A|^2 \frac{1}{a}\tanh(ax)\Big|_{-\infty}^{\infty} = \frac{2}{a}|A|^2 \Longrightarrow \boxed{A = \sqrt{\frac{a}{2}}.}$$



(c)

$$\frac{d\psi_k}{dx} = \frac{A}{ik+a}\left[(ik - a\tanh ax)ik - a^2\operatorname{sech}^2 ax\right]e^{ikx}.$$

$$\frac{d^2\psi_k}{dx^2} = \frac{A}{ik+a}\left\{ik\left[(ik - a\tanh ax)ik - a^2\operatorname{sech}^2 ax\right] - a^2 ik\operatorname{sech}^2 ax + 2a^3\operatorname{sech}^2 ax\tanh ax\right\}e^{ikx}.$$

$$
\begin{aligned}
-\frac{\hbar^2}{2m}\frac{d^2\psi_k}{dx^2}+V\psi_k &= \frac{A}{ik+a}\left\{\frac{-\hbar^2 ik}{2m}\left[-k^2-iak\tanh ax-a^2\operatorname{sech}^2 ax\right]+\frac{\hbar^2a^2}{2m}ik\operatorname{sech}^2 ax\right.\\
&\quad\left.-\frac{\hbar^2a^3}{m}\operatorname{sech}^2 ax\tanh ax-\frac{\hbar^2a^2}{m}\operatorname{sech}^2 ax(ik-a\tanh ax)\right\}e^{ikx}\\
&= \frac{Ae^{ikx}}{ik+a}\frac{\hbar^2}{2m}\left(ik^3-ak^2\tanh ax+ia^2k\operatorname{sech}^2 ax+ia^2k\operatorname{sech}^2 ax\right.\\
&\quad\left.-2a^3\operatorname{sech}^2 ax\tanh ax-2ia^2k\operatorname{sech}^2 ax+2a^3\operatorname{sech}^2 ax\tanh ax\right)\\
&= \frac{Ae^{ikx}}{ik+a}\frac{\hbar^2}{2m}k^2(ik-a\tanh ax)=\frac{\hbar^2k^2}{2m}\psi_k=E\psi_k.\quad \text{QED}
\end{aligned}
$$

As $x\to+\infty$, $\tanh ax\to+1$, so $\boxed{\psi_k(x)\to A\left(\frac{ik-a}{ik+a}\right)e^{ikx},}$ which represents a transmitted wave.

$$
\boxed{R=0.}\qquad T=\left|\frac{ik-a}{ik+a}\right|^2=\left(\frac{-ik-a}{-ik+a}\right)\left(\frac{ik-a}{ik+a}\right)=\boxed{1.}
$$