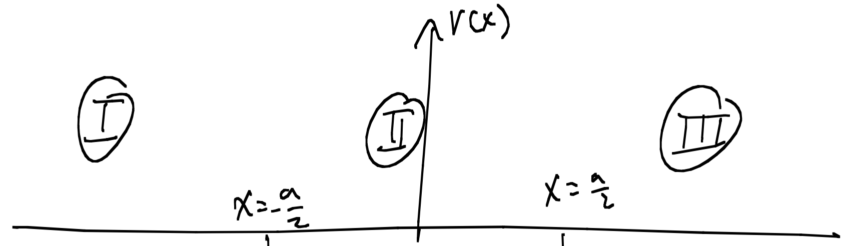
$$\psi'' = -\frac{2mE}{\hbar^2}\psi$$

E<0 (bound state)

$$\psi = C_1 e^{\kappa x} + C_2 e^{-\kappa x} \qquad \left(\kappa \equiv \sqrt{\frac{2mE}{\hbar^2}}\right)$$



$\psi_I = Ae^{\kappa x}$ $\qquad$ $\psi_{II} = B(e^{-\kappa x}+e^{\kappa x})$ $\qquad$ $\psi_{III} = Ae^{-\kappa x}$

$$\psi_I'\Big|_{x=-\frac{a}{2}} = A\kappa e^{-\frac{\kappa a}{2}}$$

$$\psi_{II}'\Big|_{x=-\frac{a}{2}} = B\kappa\left(e^{-\frac{\kappa a}{2}} - e^{\frac{\kappa a}{2}}\right)$$

$$\psi_{III}'\Big|_{x=\frac{a}{2}} = -A\kappa e^{-\frac{\kappa a}{2}}$$

$$\psi_{II}'\Big|_{x=+\frac{a}{2}} = B\kappa\left(e^{\frac{\kappa a}{2}} - e^{-\frac{\kappa a}{2}}\right)$$

ψ continuous:

$x=\frac{a}{2}$: $\quad Ae^{-\frac{\kappa a}{2}} = B\left(e^{-\frac{\kappa a}{2}} + e^{\frac{\kappa a}{2}}\right)$

$x=-\frac{a}{2}$: $\quad Ae^{-\frac{\kappa a}{2}} = B\left(e^{\frac{\kappa a}{2}} + e^{-\frac{\kappa a}{2}}\right)$

ψ' discontinuous:

$x=\frac{a}{2}$: $\quad -A\kappa e^{-\frac{\kappa a}{2}} - B\kappa\left(e^{\frac{\kappa a}{2}} - e^{-\frac{\kappa a}{2}}\right) = -\frac{2m}{\hbar^2}\frac{\hbar^2}{ma}Ae^{-\frac{\kappa a}{2}}$

$$-\kappa \cancel{B}\left(e^{\frac{\kappa a}{2}} + e^{-\frac{\kappa a}{2}}\right) - \cancel{B}\kappa\left(e^{\frac{\kappa a}{2}} - e^{-\frac{\kappa a}{2}}\right) = -\frac{2}{a}\cancel{B}\left(e^{\frac{\kappa a}{2}} + e^{-\frac{\kappa a}{2}}\right)$$

$$\kappa a = 1 + e^{-\kappa a}$$

```
clear
close all

c=2.998e10;%cm/s
hbar=6.582e-16; %in eV*sec
m=5.11e5/c^2; %in eV/c^2

dx=1e-9; %0.1 Ang, in cm

tx=hbar^2/(2*m*dx^2);

N=700+2; %size of matrix

a=1e-7; %1nm separation between deltas

WELLDEPTH=hbar^2/(m*a*dx);

V= -WELLDEPTH*[zeros(300,1);1; zeros(round(a/dx),1);1; zeros(300,1)];

H=diag(V)+tx*(diag(2*ones(N,1))+diag(-1*ones(N-1,1),1)+diag(-1*ones(N-1,1),-1));

[v,d]=eig(H);

E1=min(diag(d));

x=(1:N)*dx;
plot(x*1e7,conj(v(:,1)).*v(:,1)/dx,['*;E=' num2str(E1) 'eV;'])

xlabel('distance [nm]');
ylabel('\Psi * \Psi [cm^{-1}]')
title(['Approximated DOUBLE delta potential, \alpha=' \
      num2str(WELLDEPTH*dx*1e7) 'nm*eV, separation a=' num2str(a*1e7) 'nm'])
```

```
%solve transcendental eqn via binary search
ka_up=5;
ka_lo=0;
ka=1;
ii=0;
THRESHDIGITS=7;
do
if (1+exp(-ka)-ka)>0
  ka_lo=ka;
elseif (1+exp(-ka)-ka)<0
  ka_up=ka;
end
ka=mean([ka_up ka_lo]);
ii=ii+1;
until abs(1+exp(-ka)-ka)<10^(-THRESHDIGITS)
disp(['found kappa*a=' num2str(ka) ' to ' int2str(THRESHDIGITS) ' digits in '
int2str(ii) ' iterations'])
disp(['compare to calculated kappa*a=' num2str(sqrt(-2*m*E1/hbar^2)*a) ])

hold on;

kappa=ka/a;
plot(0:0.1:3, max(abs(v(:,1)))^2/dx*exp(2*kappa*(-3:0.1:0)*1e-7),'r;analytic;')
plot(4:0.1:7, max(abs(v(:,1)))^2/dx*exp(-2*kappa*(0:0.1:3)*1e-7),'r')
plot((3:0.1:4),max(abs(v(:,1)))^2/dx*((exp(-kappa*1e-7*(0:0.1:1))+exp(kappa*1e-
7*(-1:0.1:0)))/(1+exp(-kappa*a))).^2,'r');
```

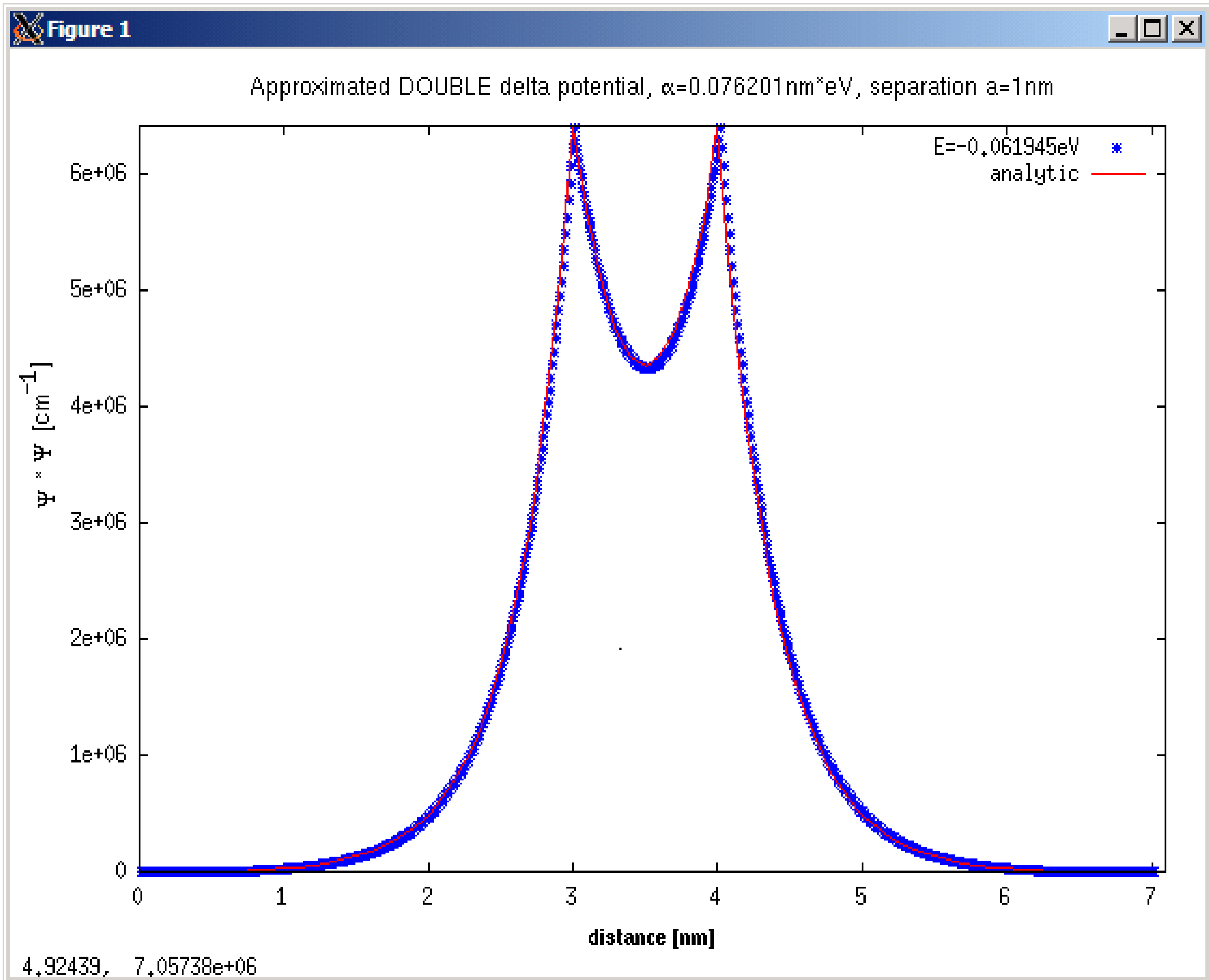


Figure 1
Approximated DOUBLE delta potential, α=0.076201nm*eV, separation a=1nm
E=-0.061945eV
analytic
6e+06
5e+06
4e+06
3e+06
2e+06
1e+06
0
Ψ*Ψ [cm^-1]
0
1
2
3
4
5
6
7
distance [nm]
4.92439, 7.05738e+06

Problem 2.24

(a) Let $y \equiv cx$, so $dx = \frac{1}{c}dy$. $\left\{\begin{array}{l}\text{If } c>0,\ y: -\infty \to \infty. \\ \text{If } c<0,\ y: \infty \to -\infty.\end{array}\right\}$

$$\int_{-\infty}^{\infty} f(x)\delta(cx)dx = \begin{cases} \frac{1}{c}\int_{-\infty}^{\infty} f(y/c)\delta(y)dy = \frac{1}{c}f(0) \quad (c>0);\ \text{or} \\ \frac{1}{c}\int_{\infty}^{-\infty} f(y/c)\delta(y)dy = -\frac{1}{c}\int_{-\infty}^{\infty} f(y/c)\delta(y)dy = -\frac{1}{c}f(0)\ (c<0). \end{cases}$$

In either case, $\int_{-\infty}^{\infty} f(x)\delta(cx)dx = \frac{1}{|c|}f(0) = \int_{-\infty}^{\infty} f(x)\frac{1}{|c|}\delta(x)dx$. So $\delta(cx) = \frac{1}{|c|}\delta(x)$. ✓

(b)

$$\int_{-\infty}^{\infty} f(x)\frac{d\theta}{dx}dx = f\theta\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{df}{dx}\theta dx \quad \text{(integration by parts)}$$

$$= f(\infty) - \int_{0}^{\infty}\frac{df}{dx}dx = f(\infty) - f(\infty) + f(0) = f(0) = \int_{-\infty}^{\infty} f(x)\delta(x)dx.$$

So $d\theta/dx = \delta(x)$. ✓ [Makes sense: The θ function is constant (so derivative is zero) except at $x = 0$, where the derivative is infinite.]

Problem 2.25

$$\psi(x) = \frac{\sqrt{m\alpha}}{\hbar}e^{-m\alpha|x|/\hbar^2} = \frac{\sqrt{m\alpha}}{\hbar}\begin{cases} e^{-m\alpha x/\hbar^2}, & (x \geq 0), \\ e^{m\alpha x/\hbar^2}, & (x \leq 0). \end{cases}$$

$$\langle x\rangle = 0 \text{ (odd integrand).}$$

$$\langle x^2\rangle = \int_{-\infty}^{\infty} x^2|\psi|^2 dx = 2\frac{m\alpha}{\hbar^2}\int_0^{\infty} x^2 e^{-2m\alpha x/\hbar^2}dx = \frac{2m\alpha}{\hbar^2}2\left(\frac{\hbar^2}{2m\alpha}\right)^3 = \frac{\hbar^4}{2m^2\alpha^2}; \quad \sigma_x = \frac{\hbar^2}{\sqrt{2}m\alpha}.$$

$$\frac{d\psi}{dx} = \frac{\sqrt{m\alpha}}{\hbar}\left\{\begin{array}{ll} -\frac{m\alpha}{\hbar^2}e^{-m\alpha x/\hbar^2}, & (x \geq 0) \\ \frac{m\alpha}{\hbar^2}e^{m\alpha x/\hbar^2}, & (x \leq 0) \end{array}\right\} = \left(\frac{\sqrt{m\alpha}}{\hbar}\right)^3\left[-\theta(x)e^{-m\alpha x/\hbar^2} + \theta(-x)e^{m\alpha x/\hbar^2}\right].$$

$$\frac{d^2\psi}{dx^2} = \left(\frac{\sqrt{m\alpha}}{\hbar}\right)^3\left[-\delta(x)e^{-m\alpha x/\hbar^2} + \frac{m\alpha}{\hbar^2}\theta(x)e^{-m\alpha x/\hbar^2} - \delta(-x)e^{m\alpha x/\hbar^2} + \frac{m\alpha}{\hbar^2}\theta(-x)e^{m\alpha x/\hbar^2}\right]$$

$$= \left(\frac{\sqrt{m\alpha}}{\hbar}\right)^3\left[-2\delta(x) + \frac{m\alpha}{\hbar^2}e^{-m\alpha|x|/\hbar^2}\right].$$

In the last step I used the fact that $\delta(-x) = \delta(x)$ (Eq. 2.142), $f(x)\delta(x) = f(0)\delta(x)$ (Eq. 2.112), and $\theta(-x) + \theta(x) = 1$ (Eq. 2.143). Since $d\psi/dx$ is an odd function, $\langle p\rangle = 0$.

$$\langle p^2\rangle = -\hbar^2\int_{-\infty}^{\infty}\psi\frac{d^2\psi}{dx^2}dx = -\hbar^2\frac{\sqrt{m\alpha}}{\hbar}\left(\frac{\sqrt{m\alpha}}{\hbar}\right)^3\int_{-\infty}^{\infty} e^{-m\alpha|x|/\hbar^2}\left[-2\delta(x) + \frac{m\alpha}{\hbar^2}e^{-m\alpha|x|/\hbar^2}\right]dx$$

$$= \left(\frac{m\alpha}{\hbar}\right)^2\left[2 - 2\frac{m\alpha}{\hbar^2}\int_0^{\infty} e^{-2m\alpha x/\hbar^2}dx\right] = 2\left(\frac{m\alpha}{\hbar}\right)^2\left[1 - \frac{m\alpha}{\hbar^2}\frac{\hbar^2}{2m\alpha}\right] = \left(\frac{m\alpha}{\hbar}\right)^2.$$

Evidently

$$\sigma_p = \frac{m\alpha}{\hbar}, \quad \text{so} \quad \sigma_x\sigma_p = \frac{\hbar^2}{\sqrt{2}m\alpha}\frac{m\alpha}{\hbar} = \sqrt{2}\frac{\hbar}{2} > \frac{\hbar}{2}. \ ✓$$

Problem 2.39

(a) According to Eq. 2.36, the most general solution to the time-dependent Schrödinger equation for the infinite square well is

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n \psi_n(x) e^{-i(n^2\pi^2\hbar/2ma^2)t}.$$

Now $\frac{n^2\pi^2\hbar}{2ma^2}T = \frac{n^2\pi^2\hbar}{2ma^2}\frac{4ma^2}{\pi\hbar} = 2\pi n^2$, so $e^{-i(n^2\pi^2\hbar/2ma^2)(t+T)} = e^{-i(n^2\pi^2\hbar/2ma^2)t}e^{-i2\pi n^2}$, and since n^2 is an integer, $e^{-i2\pi n^2} = 1$. Therefore $\Psi(x,t+T) = \Psi(x,t)$. QED

(b) The classical revival time is the time it takes the particle to go down and back: $T_c = 2a/v$, with the velocity given by

$$E = \frac{1}{2}mv^2 \Rightarrow v = \sqrt{\frac{2E}{m}} \Rightarrow \boxed{T_c = a\sqrt{\frac{2m}{E}}.}$$

(c) The two revival times are equal if

$$\frac{4ma^2}{\pi\hbar} = a\sqrt{\frac{2m}{E}}, \quad \text{or} \quad \boxed{E = \frac{\pi^2\hbar^2}{8ma^2} = \frac{E_1}{4}.}$$