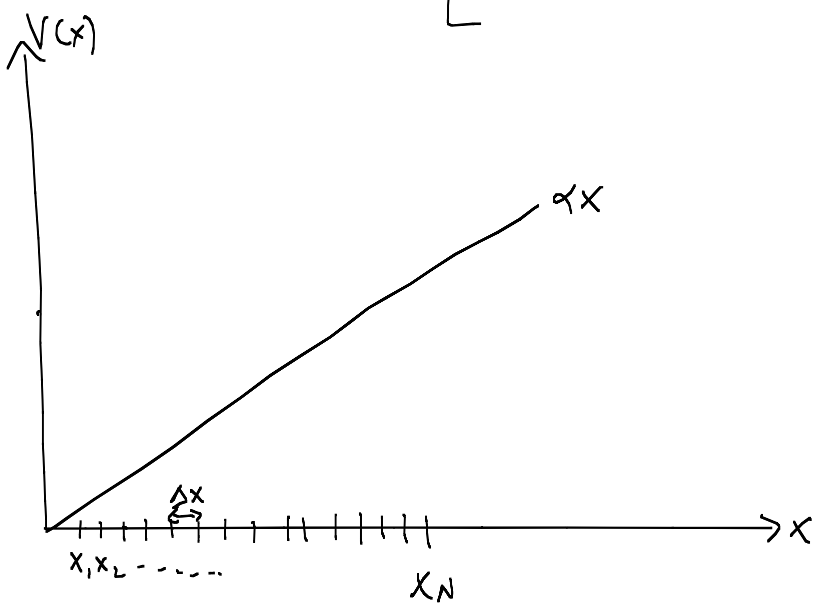
1.

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right]\varphi = E\varphi$$



$$\begin{bmatrix} \frac{\hbar^2}{m\Delta x^2}+V(x_1) & -\frac{\hbar^2}{2m\Delta x^2} & 0 & 0 & \cdots \\ -\frac{\hbar^2}{2m\Delta x^2} & \frac{\hbar^2}{m\Delta x^2}+V(x_2) & -\frac{\hbar^2}{2m\Delta x^2} & 0 & \\ & & & \ddots & \\ & & & & \ddots \end{bmatrix} \begin{bmatrix} \varphi(x_1) \\ \varphi(x_2) \\ \vdots \\ \varphi(x_N) \end{bmatrix} = E \begin{bmatrix} \varphi(x_1) \\ \varphi(x_2) \\ \vdots \\ \varphi(x_N) \end{bmatrix}$$

```
clear
close all;

c=2.998e10;%cm/s
hbar=6.582e-16; %in eV*sec
m=5.11e5/c^2; %in eV/c^2

dx=1e-9; %0.1 Ang, in cm
tx=hbar^2/(2*m*dx^2);

N=100; %size of matrix

x=dx*(1:N);
V=1e9*x; %half-triangle potential

H=tx*(diag(2*ones(N,1))+diag(-1*ones(N-1,1),1)+diag(-1
*ones(N-1,1),-1))+diag(V);

[v,d]=eig(H);

psi1=v(:,1);
psi2=v(:,2);
psi3=v(:,3);
```

plot

```
figure(1);
plot(x*1e7,V,'k;potential V(x);')
xlabel('distance [nm]');
ylabel('potential energy [eV]')
title('Half-triangular well: \alpha=10^9 eV/cm')

figure(2)
plot(x*1e7, conj(psi1).*psi1,'k;1st;')
hold on
plot(x*1e7, conj(psi2).*psi2,'r;2nd;')
plot(x*1e7, conj(psi3).*psi3,'b;3nd;')
hold off
```

plot

```
title('Half-triangular well, \alpha=10^9 eV/cm: electron density distributions')
xlabel('distance [nm]');
ylabel('\Psi * \Psi')

figure(3)
plot(diag(d),'*');
xlabel('quantum number n')
ylabel('Energy eigenvalue [eV]')
title('Half-triangular well, \alpha=10^9 eV/cm: energy eigenvalues')
axis([0 10 0 100])
```

$\langle x \rangle$:
$\langle x^2 \rangle$:
Δx:

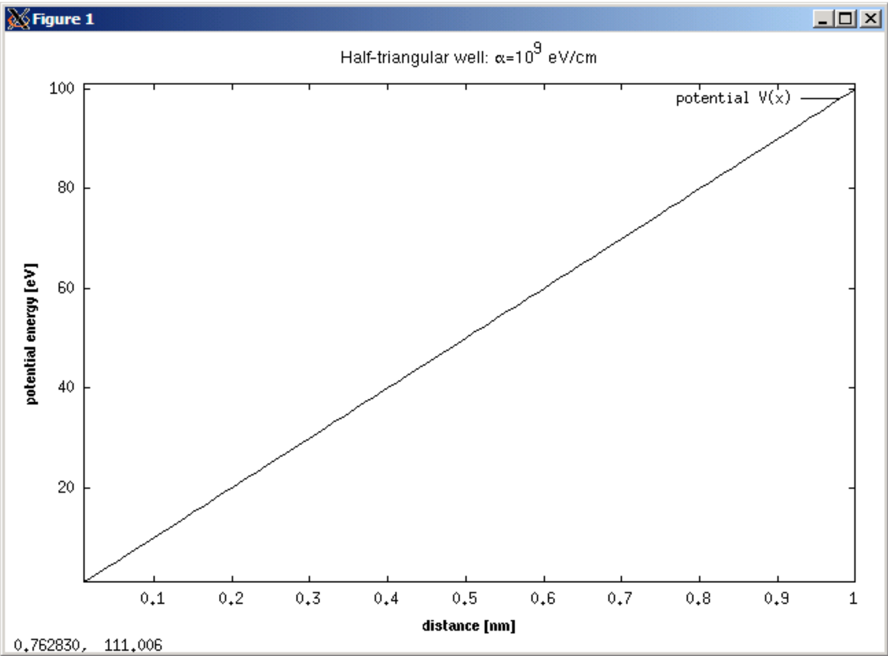
```
expct_x=psi1'*diag(x)*psi1;
expct_xsqrd=psi1'*diag(x.^2)*psi1;
Dx=sqrt(expct_xsqrd-expct_x^2);
```

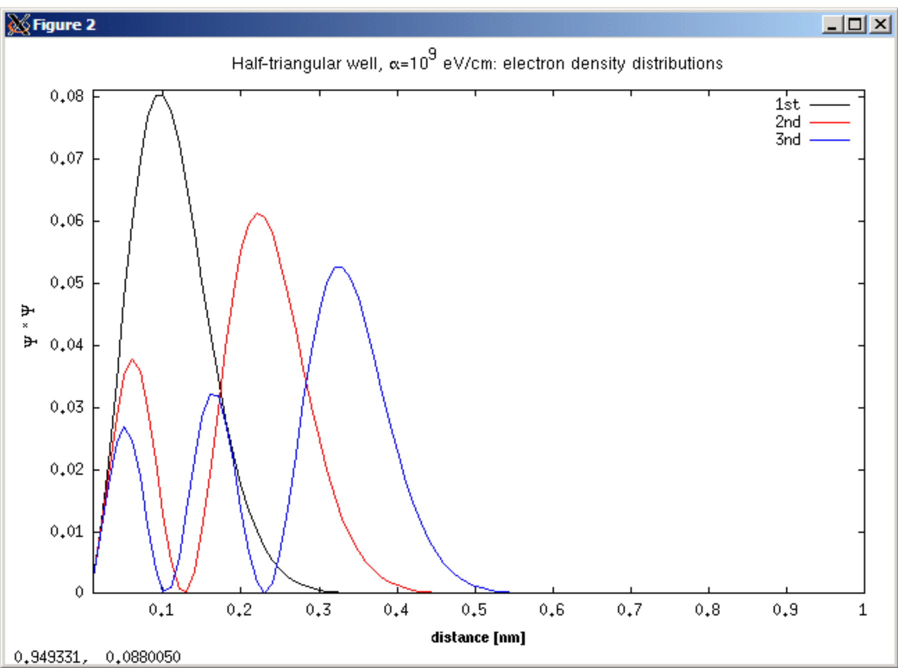
$\frac{\hbar}{i}\frac{d}{dx}$:
$-\hbar^2\frac{d^2}{dx^2}$:
$\langle p \rangle$:
$\langle p^2 \rangle$:
Δp:

```
p=hbar/i*(diag(ones(length(psi1)-1,1),1)+diag(-ones(length(psi1)-1,1),-1))/(2*dx);
psqrd=-hbar^2*(diag(-2*ones(length(psi1),1))+diag(ones(length(psi1)-1,1), 1)+diag(ones
(length(psi1)-1,1), -1))/dx^2;
expct_p=psi1'*p*psi1;
expct_psqrd=psi1'*psqrd*psi1;
Dp=sqrt(expct_psqrd-expct_p^2);

disp(['<x>=' num2str(expct_x*1e7) ' nm'])
disp(['<x^2>=' num2str(expct_xsqrd*1e14) ' nm^2'])
disp(['<p>=' num2str(expct_p*1e-7) ' nm^-1'])
disp(['<p^2>=' num2str(expct_psqrd*1e-14) ' nm^-2'])

disp(['Delta_x=' num2str(Dx*1e7) ' nm'])
disp(['Delta_p=' num2str(Dp*1e-7) ' nm^-1'])
```

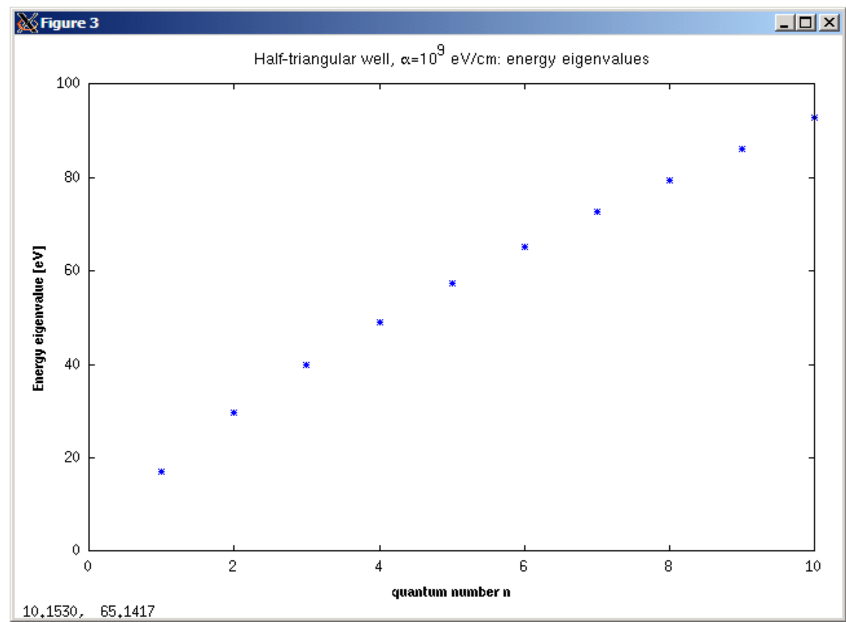
$\Delta x \Delta p \geq \frac{\hbar}{2}$:

```
disp(['Heisenberg: Dx*Dp (in units of hbar/2):' num2str(2*Dx*Dp/hbar) ' (should be > 1)']);
```





octave:1> triangleSE
<x>=0.11283 nm
<x^2>=0.015281 nm^2
<p>=0-1.3566e-31i nm^-1 ≈0
<p^2>=6.4293e-29 nm^-2
Delta_x=0.050496 nm
Delta_p=8.0183e-15 nm^-1
Heisenberg: Dx*Dp (in units of hbar/2):1.2303 (should be > 1)



2. i) $\int_{-\infty}^{\infty} \Psi^* \Psi \, dx = \int_0^a A^2 \left(\cancelto{1}{\psi_1^* \psi_1} + \cancelto{1}{\psi_2^* \psi_2} + \cancelto{0}{\psi_1^* \psi_2} + \cancelto{0}{\psi_2^* \psi_1} \right) dx$

ψ_n's are orthonormal

$$= A^2 \cdot 2 = 1 \qquad A = \sqrt{\tfrac{1}{2}}$$

ii) $\Psi(x,t) = \frac{1}{\sqrt{2}} \left(\varphi_1(x) e^{-i \frac{E_1}{\hbar} t} + \varphi_2(x) e^{-i \frac{E_2}{\hbar} t} \right)$

$$|\Psi(x,t)|^2 = \Psi^*(x,t)\, \Psi(x,t)$$

$$= \frac{1}{2} \left[\varphi_1^*(x) \varphi_1(x) + \varphi_2^* \varphi_2 + \varphi_1^* \varphi_2 \, e^{i \frac{(E_1 - E_2)t}{\hbar}} + \varphi_2^* \varphi_1 \, e^{-i \frac{(E_1 - E_2)}{\hbar} t} \right]$$

$$= \frac{1}{2} \left[\varphi_1^2(x) + \varphi_2^2(x) + 2 \varphi_1(x) \varphi_2(x) \cos \frac{E_1 - E_2}{\hbar} t \right]$$

iii) $\langle x \rangle = \int_0^a \Psi^*(x,t)\, x\, \Psi(x,t)\, dx$

$$= \int_0^a \frac{x}{2} \left[\varphi_1^2(x) + \varphi_2^2(x) + 2 \varphi_1(x) \varphi_2(x) \cos \frac{E_1 - E_2}{\hbar} t \right] dx$$

$$= \underbrace{\frac{1}{2}\left(\langle x\rangle_1 + \langle x\rangle_2\right)}_{\text{avg of } \langle x\rangle \text{ in state 1 and 2.}} + \left[\int_0^a x\, \varphi_1(x)\, \varphi_2(x)\, dx\right] \cos \frac{E_1 - E_2}{\hbar} t$$

$$\left[\int_0^a x\, \varphi_1(x)\, \varphi_2(x)\, dx\right] = \int_0^a x \sqrt{\frac{2}{a}} \sin \frac{\pi x}{a} \sqrt{\frac{2}{a}} \sin \frac{2\pi x}{a}\, dx = \int_0^a \frac{x}{a}\left(\cos \frac{\pi x}{a} - \cos \frac{3\pi x}{a}\right) dx$$

Use derivation under integral sign:

$$\int x \cos \beta x\, dx = \int \frac{d}{d\beta} \sin \beta x\, dx = \frac{d}{d\beta} \int \sin \beta x\, dx = \frac{d}{d\beta}\left(-\frac{\cos \beta x}{\beta}\right) = \frac{\cos \beta x + \beta x \sin \beta x}{\beta^2}$$

$$\left[\int_0^a x\, \varphi_1(x)\, \varphi_2(x)\, dx\right] = \frac{1}{a}\left[\frac{\cos \frac{\pi x}{a} + \frac{\pi x}{a} \sin \frac{\pi x}{a}}{\frac{\pi^2}{a^2}} - \frac{\cos \frac{3\pi x}{a} + \frac{3\pi x}{a} \sin \frac{3\pi x}{a}}{\frac{9\pi^2}{a^2}}\right]\Bigg|_0^a = -\frac{16a}{9\pi^2}$$

So $$\langle x\rangle = \frac{1}{2}\left(\langle x\rangle_1 + \langle x_2\rangle\right) - \underbrace{\frac{16a}{9\pi^2}}_{\text{amplitude}} \cos \underbrace{\frac{E_1 - E_2}{\hbar}}_{\text{angular freq.}} t$$

iv) $\Psi(x,t) = A\left(\varphi_1(x)\, e^{-i\frac{E_1}{\hbar}t} + \varphi_2(x)\, e^{-i\left(\frac{E_2}{\hbar}t - \phi\right)} \right)$, where $A = \frac{1}{\sqrt{2}}$

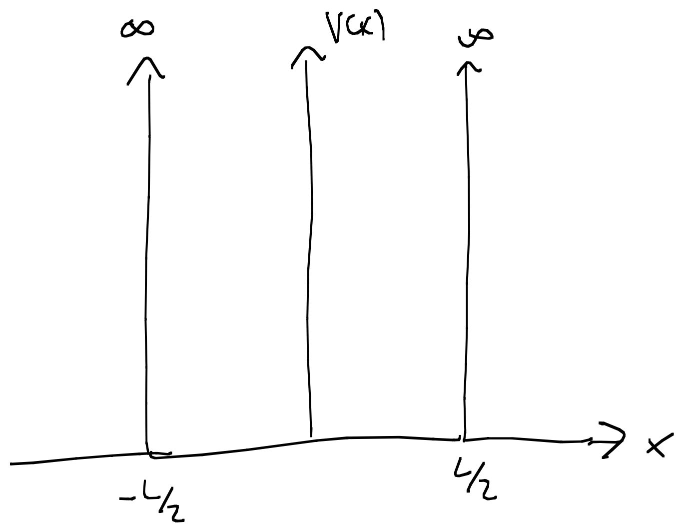
v) $|\Psi(x,t)|^2 = \Psi^*(x,t)\, \Psi(x,t)$

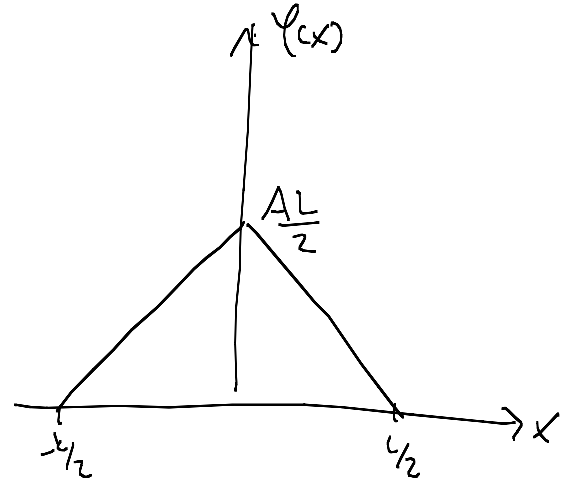
$$= \frac{1}{2}\left[\varphi_1^*(x)\varphi_1(x) + \varphi_2^*\varphi_2 + \varphi_1^*\varphi_2\, e^{i\left[\left(\frac{E_1-E_2}{\hbar}\right)t - \phi\right]} + \varphi_2^*\varphi_1\, e^{-i\left[\left(\frac{E_1-E_2}{\hbar}\right)t - \phi\right]} \right]$$

$$= \frac{1}{2}\left[\varphi_1^2(x) + \varphi_2^2(x) + 2\varphi_1(x)\varphi_2(x)\cos\left(\frac{E_1-E_2}{\hbar}t - \phi\right) \right]$$

vi) Likewise, oscillations in $\langle x \rangle$ will be phase-shifted by ϕ.

3)





(a) $[\varphi] = \frac{1}{\sqrt{\text{distance}}}$ $\left(\int_{-\infty}^{\infty} \varphi^* \varphi \, dx = 1 \right)$

$[A] \cdot \text{distance} = \frac{1}{\sqrt{\text{distance}}} \Rightarrow [A] = \text{distance}^{-3/2}$

(b) $$\int_{-L/2}^{L/2} \varphi^* \varphi \, dx = 2 \int_0^{L/2} A^2 \left(\frac{L}{2} - x\right)^2 dx = 2A^2 \int_0^{L/2} \left(\frac{L^2}{4} - Lx + x^2\right) dx$$

$$= 2A^2 \left(\frac{L^2 x}{4} - L\frac{x^2}{2} + \frac{x^3}{3}\right)\Big|_0^{L/2}$$

$$= 2A^2\left(\cancel{\frac{L^3}{8}} - \cancel{\frac{L^3}{8}} + \frac{L^3}{3\cdot 8}\right) = \frac{A^2L^3}{12} = 1$$

$$A = \sqrt{\frac{12}{L^3}} = \frac{2\sqrt{3}}{L^{3/2}}$$

(c) $a_n = 2\int_0^{L/2} A\left(\frac{L}{2} - x\right)\sqrt{\frac{2}{L}}\cos\frac{n\pi x}{L}\,dx$ $\qquad n = 1, 3, \ldots$

Wolfram Alpha says:

$$A\int_0^{\frac{L}{2}} 2\left(\frac{L}{2} - x\right)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{4L^2\overbrace{\sin^2\left(\frac{\pi n}{4}\right)}^{1/2}}{\pi^2 n^2}A = A\sqrt{\frac{2}{L}}\,\frac{4L^2\cdot\frac{1}{2}}{\pi^2n^2} = \frac{2\sqrt{3}}{\cancel{L^{3/2}}}\,\frac{\sqrt{2}\,2\,\cancel{L^{3/2}}}{\pi^2n^2} = \frac{4\sqrt{6}}{\pi^2n^2}$$

(d) $E_3 = \frac{\hbar^2\pi^2 3^2}{2ma^2} = \frac{(6.6\times10^{-16}\,eV\cdot s)^2\,\pi^2\,3^2}{2\;5.11\times10^5\,eV/(3\times10^{10}\,cm/s)^2\,(10^{-7}\,cm)^2} = 3.38\,eV$

probability $= |a_3|^2 = \left(\frac{8\sqrt{6}}{\pi^2 3^2}\right)^2$