1. Using the Fourier Transform, find the solution f(x,t) of the "drift-diffusion equation" subject to initial conditions f(x,t=0)=δ(x):

$$\frac{df}{dt} = D\frac{d^2 f}{dx^2} - v\frac{df}{dx}$$

Where D (diffusion coefficient) and v (velocity) are positive constants. Interpret the result.

Subst. $f(x,t) = \int \tilde{f}(k,t)\frac{e^{ikx}}{\sqrt{2\pi}}dk$, Then project onto orthonormal basis:

$$\frac{\partial \tilde{f}}{\partial t} = -(Dk^2 + ikv)\tilde{f} \Rightarrow \tilde{f}(k,t) = A(k)e^{-(Dk^2+ikv)t}$$

If initial conditions are $f(x,0) = \delta(x)$, $A(k) = \frac{1}{\sqrt{2\pi}}$

So

$$f(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-Dk^2t} e^{-ikvt} \frac{e^{ikx}}{\sqrt{2\pi}} dk$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-Dk^2t} e^{ik(x-vt)} dk$$

This is the same as in the notes, except $x \to x - vt$!
So we know how the integral turns out:

$$f(x,t) = \frac{1}{2\sqrt{\pi D t}} e^{-\frac{(x-vt)^2}{4Dt}}$$

(a spreading gaussian rigidly moving to the right at speed v!)

2. Hermitian matrices:

a. Write a MATLAB/octave **function** which returns a random, complex-valued **hermitian** matrix of arbitrary size *N*.

One solution is a text file called hermitian.m with contents:

```
function H=hermitian(N)
 r=rand(N)+i*rand(N);
 H=r+r';
```

Example output:

```
>> hermitian(3)

ans =

   1.6294             1.8192 + 0.7996i   0.4055 - 0.8287i
   1.8192 - 0.7996i   1.2647             0.6444 - 0.3785i
   0.4055 + 0.8287i   0.6444 + 0.3785i   1.9150
```

b. Demonstrate that the eigenvalues are real by executing the function several times and evaluating the imaginary part (for example, using `imag`()).

```
octave:19> N=100; max(imag(eig(hermitian(N))))
ans = 0
```

c. Demonstrate that the eigenvectors form a complete orthonormal set of basis functions (for instance, by calculating the inner product between several of them.)

```
octave:25>N=100;[v,d]=eig(hermitian(N));max(max(v'*v-eye(N)))
ans = -3.4417e-15
```
(this is nearly machine precision and effectively zero)

3. Using separation of variables and the method of finite differences, convert the (differential) classical wave equation

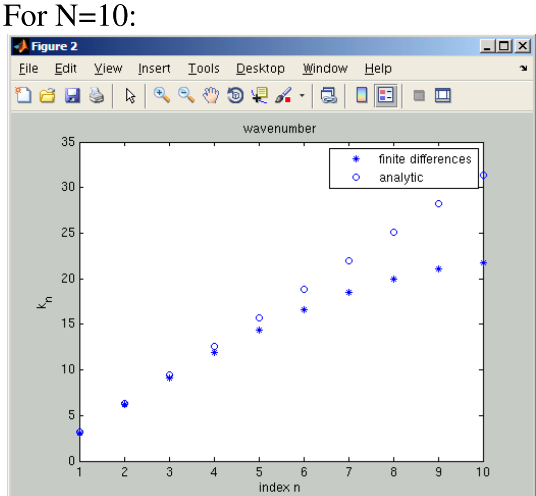
$$\frac{d^2 f}{dx^2} = \frac{1}{c^2}\frac{d^2 f}{dt^2}$$

with fixed boundary conditions *f(x=0,t)=f(x=L,t)=0* (which corresponds to a "wave on a string" between two rigid walls) into a matrix eigenvalue problem for the spatial modes.

a. Use MATLAB/octave to solve for the eigenvalues and plot the ten lowest. Compare to the analytic result.

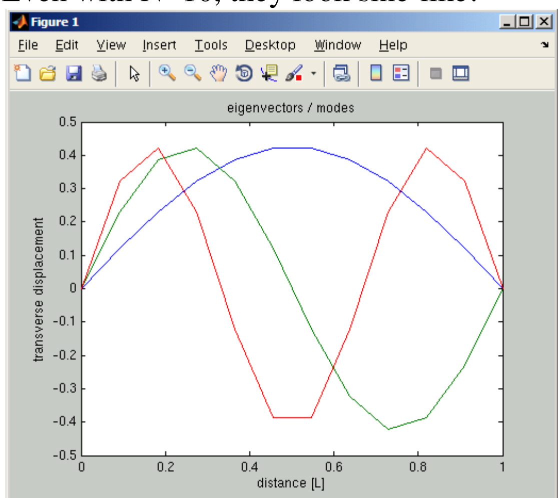
```
function ks=waveonstring(N)

L=1;
dx=L/(N+1);%discretization in x

H=diag(-2*ones(N,1))+diag(ones(N-1,1),1)+diag(ones(N-1,1),-1);

[v,d]=eig(-H); %find eigs of -H to order from lowest to highest

ks=sqrt(diag(d))/dx;

figure(1); %plot eigenvectors
NMP=3; %Number of Modes to Plot
plot([0 dx*(1:N) L],[zeros(1,NMP); v(:,1:NMP); zeros(1,NMP)]')
title('eigenvectors / modes')
ylabel('transverse displacement')
xlabel('distance [L]')


figure(2)
plot(ks,'*'); %plot numerical result for wavenumber
hold on;
plot(pi*(1:N)/L,'o'); %plot analytic wavenumber
hold off;
legend('finite differences' ,'analytic')
title('wavenumbers')
ylabel('k_n')
xlabel('index n')
```

For N=10:



b. Plot the **three** eigenvectors associated with the **three** eigenvalues having smallest absolute value. Compare to the analytic result.

Even with N=10, they look sine-like!



c. By calculating differences with the analytic wavenumber $k=n\pi/L$, show what happens to the accuracy of this approximation to eigenvalues as the size of the matrix (and the finite difference Δx) changes.

Error in, e.g. 10th eigenvalue:

```
>> ii=1;
>> for N=10:10:300
ks=waveonstring(N); err10(ii)=10*pi-ks(10); ii=ii+1;
end
>> plot(10:10:300,err10)
>> xlabel('Matric size N, L/\Delta x-1')
>> ylabel('error: numeric-analytic')
```