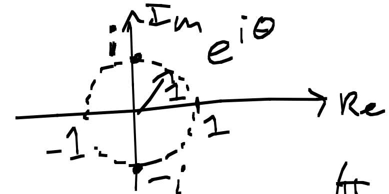
Complex numbers

(1) $i^i = \left(e^{i\frac{\pi}{2}}\right)^i = e^{i^2\frac{\pi}{2}} = e^{-\frac{\pi}{2}} = \sim 0.208$



(There are actually infinite solutions: $e^{-\left(\frac{\pi}{2}+2\pi n\right)}$ $n=0,1,2,\ldots$)

(2) $(-1)^{1/3} = \left(e^{i\pi}\right)^{1/3} = e^{i\frac{\pi}{3}} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$

$(-1)^{1/n} = e^{i\frac{\pi}{n}} = \cos\frac{\pi}{n} + i\sin\frac{\pi}{n}$

(3) $\sin^2 u = \left(\frac{e^{iu}-e^{-iu}}{2i}\right)^2 = \frac{e^{i2u}+e^{-i2u}-e^{iu}e^{-iu}-e^{-iu}e^{iu}}{-4}$

$= \frac{\frac{e^{i2u}+e^{-i2u}}{2} - \frac{2}{2}}{-2} = \frac{1-\cos 2u}{2}$

$\sin(\alpha+\beta) = \frac{e^{i(\alpha+\beta)} - e^{-i(\alpha+\beta)}}{2i} = \frac{e^{i\alpha}e^{i\beta} - e^{-i\alpha}e^{-i\beta}}{2i}$

$= \frac{2e^{i\alpha}e^{i\beta} - 2e^{-i\alpha}e^{-i\beta} + e^{i(\alpha-\beta)} + e^{-i(\alpha-\beta)} - e^{i(\alpha-\beta)} - e^{-i(\alpha-\beta)}}{4i}$

$$= \frac{e^{i\alpha} - e^{-i\alpha}}{2i}\left(\frac{e^{i\beta} + e^{-i\beta}}{2}\right) + \frac{e^{i\alpha} + e^{-i\alpha}}{2}\left(\frac{e^{i\beta} - e^{-i\beta}}{2i}\right) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

and similarly for $\sin(\alpha - \beta)$

③ $\mathrm{Re}(z)$: $\mathrm{Re}(z) = a = \frac{a+ib+a-ib}{2} = \frac{z+z^*}{2}$

$$\boxed{\begin{aligned} z &= a+ib \\ z^* &= a-ib \end{aligned}}$$

$$\mathrm{Im}(z) = b = \frac{a-a+ib-(-ib)}{2i} = \frac{a+ib-(a-ib)}{2i} = \frac{z-z^*}{2i}$$

$$\cos(z) = \frac{\cos z + \cos z + i\sin z - i\sin z}{2} = \frac{(\cos z + i\sin z) + (\cos z - i\sin z)}{2} = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sin(z) = \frac{i\sin z + i\sin z + \cos z - \cos z}{2i} = \frac{(\cos z + i\sin z) - (\cos z - i\sin z)}{2i}$$

$$= \frac{e^{iz} - e^{-iz}}{2i}$$

Fourier Transforms

1. Modulation Thm.

$$FT(F(x)\cos k_0x): \int_{-\infty}^{\infty}\frac{e^{-ikx}}{\sqrt{2\pi}}F(x)\cos k_0x\,dx$$

$$= \int_{-\infty}^{\infty}\frac{e^{-ikx}}{\sqrt{2\pi}}F(x)\,\frac{e^{ik_0x}+e^{-ik_0x}}{2}\,dx$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty}\frac{e^{-i(k-k_0)x}}{\sqrt{2\pi}}F(x)\,dx + \int_{-\infty}^{\infty}\frac{e^{-i(k+k_0)x}}{\sqrt{2\pi}}F(x)\,dx\right]$$

If $A(k) = \int_{-\infty}^{\infty}\frac{e^{ikx}}{\sqrt{2\pi}}F(x)\,dx$,

$$FT(F(x)\cos k_0x): \frac{1}{2}\left[A(k-k_0) + A(k+k_0)\right]$$

2.

(a) $$f_e(x) = \frac{f(x) + f(-x)}{2}$$

$$f_o(x) = \frac{f(x) - f(-x)}{2}$$

$$f_e(-x) = \frac{f(-x) + f(x)}{2} = \frac{f(x) + f(-x)}{2} = f_e(x) \quad \text{(even)}$$

$$f_o(-x) = \frac{f(-x) - f(x)}{2} = -\left[\frac{f(x) - f(-x)}{2}\right] = -f_o(x) \quad \text{(odd)}$$

$$f_e(x) + f_o(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2} = f(x)$$

(b) $$F(x) = \int_{-\infty}^{\infty} A(k) \frac{e^{ikx}}{\sqrt{2\pi}} dk = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) \cos kx \, dk + \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) \sin kx \, dk$$

$$A(k) = A_e(k) + A_o(k)$$

$$= \frac{1}{\sqrt{2\pi}} \left[\underbrace{\int_{-\infty}^{\infty} A_e(k)\cos kx\, dk}_{\text{even}} + \underbrace{\cancelto{0}{\int_{-\infty}^{\infty} A_o(k)\cos kx\, dk}}_{\text{odd}} + i \underbrace{\cancelto{0}{\int_{-\infty}^{\infty} A_e(k)\sin kx\, dk}}_{\text{odd}} + i \underbrace{\int_{-\infty}^{\infty} A_o(k)\sin kx\, dk}_{\text{even}} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} A_e(k)\cos kx\, dk + i \int_{-\infty}^{\infty} A_o(k)\sin kx\, dk \right]$$

$$= \frac{2}{\sqrt{2\pi}} \left[\int_{0}^{\infty} A_e(k)\cos kx\, dk + i \int_{0}^{\infty} A_o(k)\sin kx\, dk \right]$$

$$= \frac{1}{\sqrt{\pi}} \int_0^{\infty} \underbrace{\sqrt{2}\, \frac{A(k)+A(-k)}{2}}_{C(k)} \cos kx\, dk + \frac{1}{\sqrt{\pi}} \int_0^{\infty} \underbrace{\sqrt{2}\, i\, \frac{A(k)-A(-k)}{2}}_{S(k)} \sin kx\, dk$$

3. Symmetry

$$F(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(k)\, e^{ikx}\, dk = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(k)\,(\cos kx + i\sin kx)\, dk$$

<u>If $A(k)$ real:</u>

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{\infty} A_e(k)\cos kx\, dk + \cancelto{0}{\int_{-\infty}^{\infty} A_o(k)\cos kx\, dk} + i\cancelto{0}{\int_{-\infty}^{\infty} A_e(k)\sin kx\, dk} + i\int_{-\infty}^{\infty} A_o(k)\sin kx\, dk\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\overbrace{\int_{-\infty}^{\infty} A_e(k)\cos kx\, dk}^{\text{real part of } F(x)} + \overbrace{i\int_{-\infty}^{\infty} A_o(k)\sin kx\, dk}^{\text{imaginary part of } F(x)}\right]$$

For $F(x)$ to be real, $A_o(k)=0$ so $A(k) = A_e(k) = A(-k)$

For $F(x)$ to be imaginary, $A_e(k)=0$ so $A(k) = A_o(k) = -A(-k)$

<u>If $A(k)$ imaginary:</u>

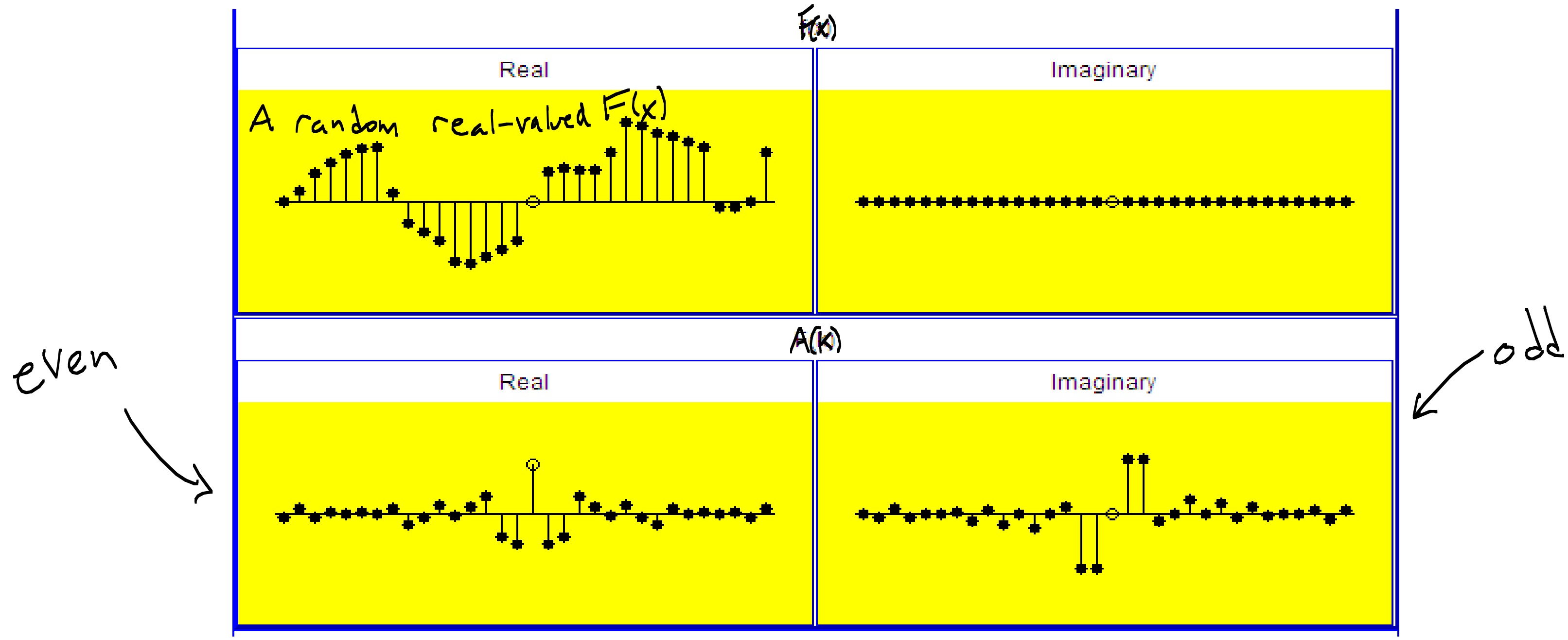
$$F(x) = \frac{1}{\sqrt{2\pi}}\left[i\int_{-\infty}^{\infty} A_e(k)\cos kx\, dk - \int_{-\infty}^{\infty} A_o(k)\sin kx\, dk\right]$$

For $F(x)$ to be real, $A_e(k)=0$ so $A(k) = A_o(k) = -A(-k)$

For $F(x)$ to be imaginary, $A_o(k)=0$ so $A(k) = A_e(k) = A(-k)$

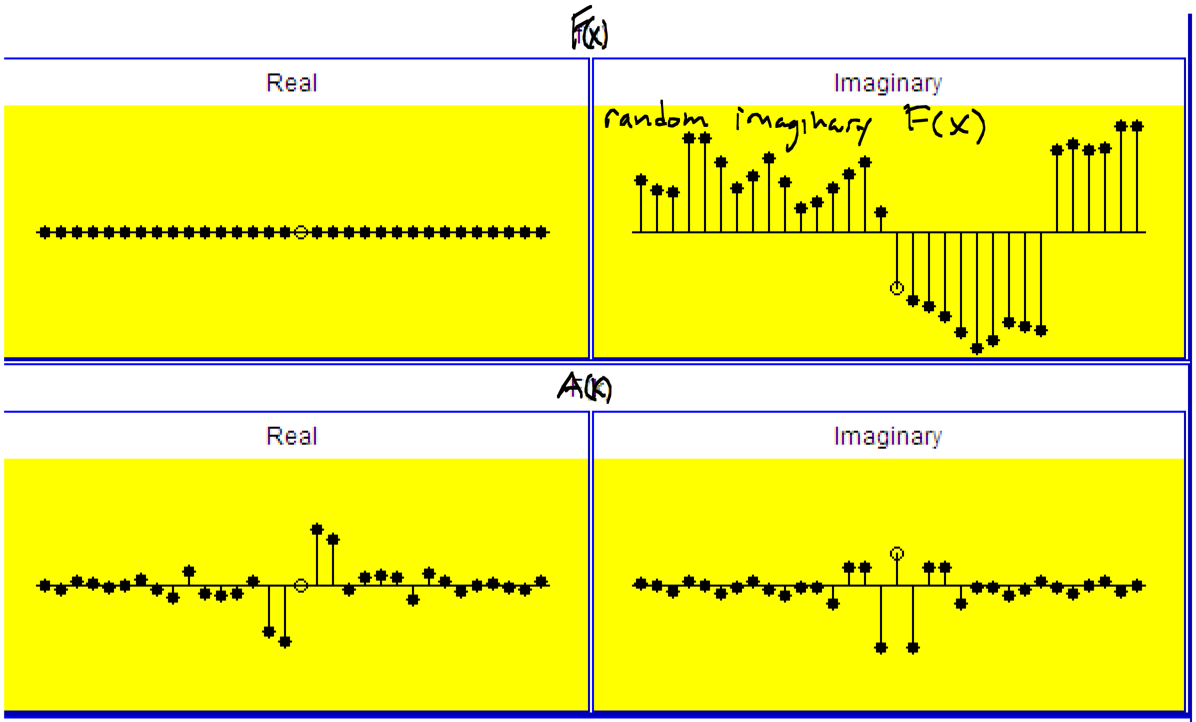
In other words,

(a) For $F(x)$ to be real, $A(K) = \overbrace{A_r(K)}^{\text{even}} + i\,\overbrace{A_i(K)}^{\text{odd}}$



(b) For $F(x)$ to be imaginary, $A(k) = \overbrace{A_r(k)}^{\text{odd}} + i\,\overbrace{A_i(k)}^{\text{even}}$

odd



even

Differential Eqns

1. $$\vec{\nabla}\cdot\vec{\mathcal{E}} = 0$$
$$\vec{\nabla}\times\vec{\mathcal{E}} = -\frac{\partial \vec{B}}{\partial t}$$
$$\vec{\nabla}\cdot\vec{B} = 0$$
$$\vec{\nabla}\times\vec{B} = \mu_0 \epsilon \frac{d\vec{\mathcal{E}}}{\partial t}$$

Maxwell's eqns

$$\vec{\nabla}\times\vec{\nabla}\times\vec{\mathcal{E}} = \vec{\nabla}\left(\cancelto{0}{\vec{\nabla}\cdot\vec{\mathcal{E}}}\right) - \nabla^2\vec{\mathcal{E}} = -\frac{\partial}{\partial t}\vec{\nabla}\times\vec{B} = -\mu_0\epsilon_0\frac{\partial^2}{\partial t^2}\vec{\mathcal{E}}$$

$$\nabla^2\vec{\mathcal{E}} = \frac{1}{c^2}\frac{\partial^2\vec{\mathcal{E}}}{\partial t^2}$$

If $\vec{\mathcal{E}}(z,t) = \mathcal{E}_x(z,t)\,\hat{x}$,

$$\frac{\partial^2\mathcal{E}_x}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2\mathcal{E}_x}{\partial t^2}$$

"wave equation"

2. D'Alembert Solution

$$\mathcal{E}_x(z,t) = \int_{-\infty}^{\infty} A(k,t) \frac{e^{ikz}}{\sqrt{2\pi}} dk$$

wave eqn:

$$-k^2 A(k,t) = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} A(k,t)$$

$$A(k,t) = B(k) e^{\pm ikct}$$

$$\mathcal{E}_x(z,t) = \int_{-\infty}^{\infty} B(k) e^{\pm ikct} \frac{e^{ikz}}{\sqrt{2\pi}} dk = \int_{-\infty}^{\infty} B(k) \frac{e^{i(kz \pm kct)}}{\sqrt{2\pi}} dk$$

$$= \int_{-\infty}^{\infty} B(k) \frac{e^{ik(z \pm ct)}}{\sqrt{2\pi}} dk$$

$$= \mathcal{E}_x(z \pm ct)$$

3.

$$\frac{df}{dz} = Af \longrightarrow f(z) = Ce^{Az}$$

$$\frac{d^2f}{dr^2} = Af \longrightarrow f(r) = C_1 e^{\sqrt{A}\,r} + C_2 e^{-\sqrt{A}\,r}$$

$$\frac{d^2f}{dt^2} = -Af \longrightarrow f(t) = C_1 e^{i\sqrt{A}\,t} + C_2 e^{-i\sqrt{A}\,t}$$

$$\frac{d^2f}{dy^2} = 0 \longrightarrow f(y) = C_1 y + C_2$$

$$\frac{d^2f}{dx^2} = A \longrightarrow f(x) = \frac{A}{2}x^2 + C_1 x + C_2$$