Solution to Problem 4.13

Part (a)

We want the expectation value of

$$\langle r\rangle = \int_V r|\psi|^2 dV. \tag{23}$$

The ground state wavefunction is

$$\psi = \psi_{1,0,0} = Ne^{-r/a}, \quad N = \frac{1}{(\pi a^3)^{1/2}}, \quad dV = r^2 dr \sin\theta d\theta d\phi \to 4\pi r^2 dr. \tag{24}$$

Thus we get

$$\langle r\rangle = 4\pi \int_V r|\psi|^2 r^2 dr = N^2 4\pi \int_0^\infty e^{-2r/a} r^3 dr = \frac{4}{a^3}\int_0^\infty e^{-2r/a} r^3 dr. \tag{25}$$

Similarly

$$\langle r^2 \rangle = 4\pi^2 \int_V r^2 |\psi|^2 r^2 dr = \frac{4}{a^3} \int_0^\infty e^{-2r/a} r^4 dr. \tag{26}$$

We can derive this integral similarly to what was done in previous hwks. At this point however you may wish to recall:

$$\int_0^\infty e^{-kr} r^n dr = k^{-1-n} n! \quad Re(n) > -1 \;\; and \;\; Re(k) > 0. \tag{27}$$

Then with $k = 2/a$

$$\langle r \rangle = \frac{4}{a^3} \int_0^\infty e^{-k} r^3 dr = \frac{4}{a^3} \frac{6}{k^4} = \frac{4}{a^3} \frac{3a^4}{8} = \frac{3}{2} a. \tag{28}$$

Likewise,

$$\langle r^2 \rangle = \frac{4}{a^3} \int_0^\infty e^{-kr} r^4 dr = \frac{4}{a^3} \frac{24}{k^5} = \frac{4}{a^3} \frac{3a^5}{4} = 3a^2. \tag{29}$$

Part (b)

$\langle x \rangle = 0$ (use $x = r \sin\theta \cos\phi$ and note the integral over ϕ from 0 to 2π gives zero). Also by symmetry $\langle x^2 \rangle = \langle r^2 \rangle / 3 = a^2$.

Part (c)

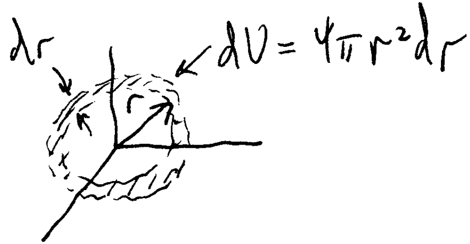
Using the tables in Griffiths, you should be able to construct (R_{21} normalized and then multiply by Y_1^1)

$$\psi_{2,1,1} = -\frac{1}{(\pi a)^{1/2}} \frac{1}{8a^2} r e^{-\frac{r}{2a}} \sin\theta e^{i\phi}. \tag{30}$$

Then using $x^2 = (r \sin\theta \cos\phi)^2$

$$\begin{aligned} \langle x^2 \rangle &= \frac{1}{64\pi a^5} \int_0^\infty r^6 e^{-r/a} dr \int_0^\pi \sin^5\theta d\theta \int_0^{2\pi} \cos^2\phi d\phi & (31) \\ &= \frac{1}{64\pi a^5} * \frac{6!}{(1/a)^7} * (16/15) * (\pi) = 12a^2. & (32) \end{aligned}$$

4.14. $|\psi(\vec{r})|^2 d^3\vec{r}$ = Prob. in volume $d^3\vec{r}$



Ground state spherically symmetric $\Rightarrow$ $\psi_{100} = \frac{1}{\sqrt{\pi a^3}} e^{-\frac{r}{a}}$

$\Rightarrow$ $P(r)dr = |\psi_{100}|^2 \cdot 4\pi r^2 dr = \frac{1}{\pi a^3} e^{-\frac{2r}{a}} \cdot 4\pi r^2 dr$

$= \frac{4}{a^3} r^2 e^{-\frac{2r}{a}} dr$

Max where $\frac{d}{dr}\left(r^2 e^{-\frac{2r}{a}}\right) = 0$

$2r e^{-\frac{2r}{a}} - \frac{2r^2}{a} e^{-\frac{2r}{a}} = 0 \Rightarrow \frac{r}{a} = 1$ or $\boxed{r = a}$

Problem 4.24

(a)

$$H = 2\left(\frac{1}{2}mv^2\right) = mv^2; \quad |\mathbf{L}| = 2\frac{a}{2}mv = amv, \quad \text{so } L^2 = a^2m^2v^2, \quad \text{and hence} \quad H = \frac{L^2}{ma^2}.$$

But we know the eigenvalues of $L^2 : \hbar^2 l(l+1)$; or, since we usually label energies with n:

$$\boxed{E_n = \frac{\hbar^2 n(n+1)}{ma^2} \quad (n = 0, 1, 2, \ldots).}$$

(b) $\boxed{\psi_{nm}(\theta, \phi) = Y_n^m(\theta, \phi),}$ the ordinary spherical harmonics. The degeneracy of the nth energy level is the number of m-values for given n: $\boxed{2n+1.}$