Problem 3.32

$$\Psi(x,t) = \frac{1}{\sqrt{2}}\left(\psi_1 e^{-iE_1t/\hbar} + \psi_2 e^{-iE_2t/\hbar}\right); \qquad \langle\Psi(x,t)|\Psi(x,0)\rangle = 0 \Rightarrow$$

$$\frac{1}{2}\left(e^{iE_1t/\hbar}\langle\psi_1|\psi_1\rangle + e^{iE_1t/\hbar}\langle\psi_1|\psi_2\rangle + e^{iE_2t/\hbar}\langle\psi_2|\psi_1\rangle + e^{iE_2t/\hbar}\langle\psi_2|\psi_2\rangle\right)$$

$$= \frac{1}{2}\left(e^{iE_1t/\hbar} + e^{iE_2t/\hbar}\right) = 0, \text{ or } e^{iE_2t/\hbar} = -e^{iE_1t/\hbar}, \text{ so } e^{i(E_2-E_1)t/\hbar} = -1 = e^{i\pi}.$$

Thus $(E_2 - E_1)t/\hbar = \pi$ (orthogonality also at 3π, 5π, etc., but this is the *first* occurrence).

$$\therefore \Delta t \equiv \frac{t}{\pi} = \frac{\hbar}{E_2 - E_1}. \quad \text{But } \Delta E = \sigma_H = \frac{1}{2}(E_2 - E_1) \text{ (Problem 3.18). } \quad \text{So } \Delta t\, \Delta E = \frac{\hbar}{2}. \quad \checkmark$$

Problem 3.37

First find the eigenvalues and eigenvectors of the Hamiltonian. The characteristic equation says

$$\begin{vmatrix} (a-E) & 0 & b \\ 0 & (c-E) & 0 \\ b & 0 & (a-E) \end{vmatrix} = (a-E)(c-E)(a-E) - b^2(c-E) = (c-E)\left[(a-E)^2 - b^2\right] = 0,$$

Either $E = c$, or else $(a-E)^2 = b^2 \Rightarrow E = a \pm b$. So the eigenvalues are

$$E_1 = c, \quad E_2 = a+b, \quad E_3 = a-b.$$

To find the corresponding eigenvectors, write

$$\begin{pmatrix} a & 0 & b \\ 0 & c & 0 \\ b & 0 & a \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = E_n \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}.$$

(1)

$$\left.\begin{aligned} a\alpha + b\gamma &= c\alpha \Rightarrow (a-c)\alpha + b\gamma = 0; \\ c\beta &= c\beta \qquad \text{(redundant)} \qquad ; \\ b\alpha + a\gamma &= c\gamma \Rightarrow (a-c)\gamma + b\alpha = 0. \end{aligned}\right\} \Rightarrow \left[(a-c)^2 - b^2\right]\alpha = 0.$$

So (excluding the degenerate case $a - c = \pm b$) $\alpha = 0$, and hence also $\gamma = 0$.

(2)

$$\begin{aligned} a\alpha + b\gamma &= (a+b)\alpha \Rightarrow & \alpha - \gamma &= 0; \\ c\beta &= (a+b)\beta \Rightarrow & \beta &= 0; \\ b\alpha + a\gamma &= (a+b)\gamma & \text{(redundant)}&. \end{aligned}$$

So $\alpha = \gamma$ and $\beta = 0$.

(3)

$$\begin{aligned} a\alpha + b\gamma &= (a-b)\alpha \Rightarrow & \alpha + \gamma &= 0; \\ c\beta &= (a-b)\beta \Rightarrow & \beta &= 0; \\ b\alpha + a\gamma &= (a-b)\gamma & \text{(redundant)}&. \end{aligned}$$

So $\alpha = -\gamma$ and $\beta = 0$.

Conclusion: The (normalized) eigenvectors of H are

$$|s_1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |s_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad |s_3\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

(a) Here $|\mathcal{S}(0)\rangle = |s_1\rangle$, so

$$|\mathcal{S}(t)\rangle = e^{-iE_1 t/\hbar}|s_1\rangle = \boxed{e^{-ict/\hbar} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.}$$

(b)

$$|\mathcal{S}(0)\rangle = \frac{1}{\sqrt{2}}\left(|s_2\rangle + |s_3\rangle\right).$$

$$|\mathcal{S}(t)\rangle = \frac{1}{\sqrt{2}}\left(e^{-iE_2 t/\hbar}|s_2\rangle + e^{-iE_3 t/\hbar}|s_3\rangle\right) = \frac{1}{\sqrt{2}}\left[e^{-i(a+b)t/\hbar}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + e^{-i(a-b)t/\hbar}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\right]$$

$$= \frac{1}{2}e^{-iat/\hbar}\begin{pmatrix} e^{-ibt/\hbar} + e^{ibt/\hbar} \\ 0 \\ e^{-ibt/\hbar} - e^{ibt/\hbar} \end{pmatrix} = \boxed{e^{-iat/\hbar}\begin{pmatrix} \cos(bt/\hbar) \\ 0 \\ -i\sin(bt/\hbar) \end{pmatrix}.}$$

Problem 3.38

(a) H:

$$\boxed{E_1 = \hbar\omega,\ E_2 = E_3 = 2\hbar\omega;\quad |h_1\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix},\ |h_2\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix},\ |h_3\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix}.}$$

A:

$$\begin{vmatrix}-a & \lambda & 0\\ \lambda & -a & 0\\ 0 & 0 & (2\lambda - a)\end{vmatrix} = a^2(2\lambda - a) - (2\lambda - a)\lambda^2 = 0 \Rightarrow \boxed{a_1 = 2\lambda,\ a_2 = \lambda,\ a_3 = -\lambda.}$$

$$\lambda\begin{pmatrix}0&1&0\\1&0&0\\0&0&2\end{pmatrix}\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix} = a\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix} \Rightarrow \begin{cases}\lambda\beta = a\alpha\\ \lambda\alpha = a\beta\\ 2\lambda\gamma = a\gamma\end{cases}$$

(1)

$$\left.\begin{aligned}\lambda\beta &= 2\lambda\alpha \Rightarrow \beta = 2\alpha,\\ \lambda\alpha &= 2\lambda\beta \Rightarrow \alpha = 2\beta,\\ 2\lambda\gamma &= 2\lambda\gamma;\end{aligned}\right\}\quad \alpha = \beta = 0;\quad \boxed{|a_1\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix}.}$$

(2)

$$\left.\begin{aligned}\lambda\beta &= \lambda\alpha \Rightarrow \beta = \alpha,\\ \lambda\alpha &= \lambda\beta \Rightarrow \alpha = \beta,\\ 2\lambda\gamma &= \lambda\gamma; \Rightarrow \gamma = 0.\end{aligned}\right\}\quad \boxed{|a_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}.}$$

(3)

$$\left.\begin{aligned}\lambda\beta &= -\lambda\alpha \Rightarrow \beta = -\alpha,\\ \lambda\alpha &= -\lambda\beta \Rightarrow \alpha = -\beta,\\ 2\lambda\gamma &= -\lambda\gamma; \Rightarrow \gamma = 0.\end{aligned}\right\}\quad \boxed{|a_3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}.}$$

B:

$$\begin{vmatrix}(2\mu - b) & 0 & 0\\ 0 & -b & \mu\\ 0 & \mu & -b\end{vmatrix} = b^2(2\mu - b) - (2\mu - b)\mu^2 = 0 \Rightarrow \boxed{b_1 = 2\mu,\ b_2 = \mu,\ b_3 = -\mu.}$$

$$\mu\begin{pmatrix}2&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix} = b\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix} \Rightarrow \begin{cases}2\mu\alpha = b\alpha\\ \mu\gamma = b\beta\\ \mu\beta = b\gamma\end{cases}$$

(1)

$$\left.\begin{aligned}2\mu\alpha &= 2\mu\alpha,\\ \mu\gamma &= 2\mu\beta \Rightarrow \gamma = 2\beta,\\ \mu\beta &= 2\mu\gamma \Rightarrow \beta = 2\gamma;\end{aligned}\right\}\quad \beta = \gamma = 0;\quad \boxed{|b_1\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}.}$$

(2)

$$\left.\begin{aligned}2\mu\alpha &= \mu\alpha \Rightarrow \alpha = 0,\\ \mu\gamma &= \mu\beta \Rightarrow \gamma = \beta,\\ \mu\beta &= \mu\gamma; \Rightarrow \beta = \gamma.\end{aligned}\right\}\quad \boxed{|b_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix}.}$$

(3)

$$\left.\begin{aligned}2\mu\alpha &= -\mu\alpha \Rightarrow \alpha = 0,\\ \mu\gamma &= -\mu\beta \Rightarrow \gamma = -\beta,\\ \mu\beta &= -\mu\gamma; \Rightarrow \beta = -\gamma.\end{aligned}\right\}\quad \boxed{|b_3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\end{pmatrix}.}$$

(b)

$$\langle H\rangle = \langle \mathcal{S}(0)|H|\mathcal{S}(0)\rangle = \hbar\omega \begin{pmatrix} c_1^* & c_2^* & c_3^* \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \boxed{\hbar\omega\left(|c_1|^2 + 2|c_2|^2 + 2|c_3|^2\right).}$$

$$\langle A\rangle = \langle \mathcal{S}(0)|A|\mathcal{S}(0)\rangle = \lambda \begin{pmatrix} c_1^* & c_2^* & c_3^* \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \boxed{\lambda\left(c_1^* c_2 + c_2^* c_1 + 2|c_3|^2\right).}$$

$$\langle B\rangle = \langle \mathcal{S}(0)|B|\mathcal{S}(0)\rangle = \mu \begin{pmatrix} c_1^* & c_2^* & c_3^* \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \boxed{\mu\left(2|c_1|^2 + c_2^* c_3 + c_3^* c_2\right).}$$

(c)

$$|\mathcal{S}(0)\rangle = c_1|h_1\rangle + c_2|h_2\rangle + c_3|h_3\rangle \quad \Rightarrow$$

$$|\mathcal{S}(t)\rangle = c_1 e^{-iE_1 t/\hbar}|h_1\rangle + c_2 e^{-iE_2 t/\hbar}|h_2\rangle + c_3 e^{-iE_3 t/\hbar}|h_3\rangle = c_1 e^{-i\omega t}|h_1\rangle + c_2 e^{-2i\omega t}|h_2\rangle + c_3 e^{-2i\omega t}|h_3\rangle$$

$$= e^{-2i\omega t}\left[c_1 e^{i\omega t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right] = \boxed{e^{-2i\omega t}\begin{pmatrix} c_1 e^{i\omega t} \\ c_2 \\ c_3 \end{pmatrix}.}$$

H: $\boxed{h_1 = \hbar\omega,\ \text{probability } |c_1|^2; \quad h_2 = h_3 = 2\hbar\omega,\ \text{probability } (|c_2|^2 + |c_3|^2).}$

A: $\boxed{a_1 = 2\lambda,}$ $\langle a_1|\mathcal{S}(t)\rangle = e^{-2i\omega t}\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} c_1 e^{i\omega t} \\ c_2 \\ c_3 \end{pmatrix} = e^{-2i\omega t} c_3 \Rightarrow \boxed{\text{probability } |c_3|^2.}$

$\boxed{a_2 = \lambda,}$ $\langle a_2|\mathcal{S}(t)\rangle = e^{-2i\omega t}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} c_1 e^{i\omega t} \\ c_2 \\ c_3 \end{pmatrix} = \frac{1}{\sqrt{2}} e^{-2i\omega t}\left(c_1 e^{i\omega t} + c_2\right) \Rightarrow$

$$\text{probability} = \frac{1}{2}\left(c_1^* e^{-i\omega t} + c_2^*\right)\left(c_1 e^{i\omega t} + c_2\right) = \boxed{\frac{1}{2}\left(|c_1|^2 + |c_2|^2 + c_1^* c_2 e^{-i\omega t} + c_2^* c_1 e^{i\omega t}\right).}$$

$\boxed{a_3 = -\lambda,}$ $\langle a_3|\mathcal{S}(t)\rangle = e^{-2i\omega t}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} c_1 e^{i\omega t} \\ c_2 \\ c_3 \end{pmatrix} = \frac{1}{\sqrt{2}} e^{-2i\omega t}\left(c_1 e^{i\omega t} - c_2\right) \Rightarrow$

$$\text{probability} = \frac{1}{2}\left(c_1^* e^{-i\omega t} - c_2^*\right)\left(c_1 e^{i\omega t} - c_2\right) = \boxed{\frac{1}{2}\left(|c_1|^2 + |c_2|^2 - c_1^* c_2 e^{-i\omega t} - c_2^* c_1 e^{i\omega t}\right).}$$

Note that the sum of the probabilities is 1.

B: $\boxed{b_1 = 2\mu,}$ $\langle b_1|\mathcal{S}(t)\rangle = e^{-2i\omega t}\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} c_1 e^{i\omega t} \\ c_2 \\ c_3 \end{pmatrix} = e^{-2i\omega t} c_1 \Rightarrow \boxed{\text{probability } |c_1|^2.}$

$\boxed{b_2 = \mu,}$ $\langle b_2|\mathcal{S}(t)\rangle = e^{-2i\omega t}\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} c_1 e^{i\omega t} \\ c_2 \\ c_3 \end{pmatrix} = \frac{1}{\sqrt{2}} e^{-2i\omega t}\left(c_2 + c_3\right) \Rightarrow$

$$\text{probability} = \frac{1}{2}\left(c_1^* + c_2^*\right)\left(c_1 + c_2\right) = \boxed{\frac{1}{2}\left(|c_1|^2 + |c_2|^2 + c_1^* c_2 + c_2^* c_1\right).}$$

$\boxed{b_3 = -\mu,}$ $\langle b_3|\mathcal{S}(t)\rangle = e^{-2i\omega t}\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} c_1 e^{i\omega t} \\ c_2 \\ c_3 \end{pmatrix} = \frac{1}{\sqrt{2}} e^{-2i\omega t}\left(c_2 - c_3\right) \Rightarrow$

$$\text{probability} = \frac{1}{2}\left(c_2^* - c_3^*\right)\left(c_2 - c_3\right) = \boxed{\frac{1}{2}\left(|c_2|^2 + |c_3|^2 - c_2^* c_3 - c_3^* c_2\right).}$$

Again, the sum of the probabilities is 1.

Problem 3.39

(a)

Expanding in a Taylor series: $f(x+x_0) = \sum_{n=0}^{\infty} \frac{1}{n!} x_0^n \left(\frac{d}{dx}\right)^n f(x)$.

But $p = \frac{\hbar}{i}\frac{d}{dx}$, so $\frac{d}{dx} = \frac{ip}{\hbar}$. Therefore $f(x+x_0) = \sum_{n=0}^{\infty} \frac{1}{n!} x_0^n \left(\frac{ip}{\hbar}\right)^n f(x) = e^{ipx_0/\hbar} f(x)$.

(b)

$$\Psi(x, t+t_0) = \sum_{n=0}^{\infty} \frac{1}{n!} t_0^n \left(\frac{\partial}{\partial t}\right)^n \Psi(x,t); \quad i\hbar\frac{\partial \Psi}{\partial t} = H\Psi.$$

[*Note:* It is emphatically *not* the case that $i\hbar\frac{\partial}{\partial t} = H$. These two operators have the same effect *only* when (as here) they are acting on solutions to the (time-dependent) Schrödinger equation.] Also,

$$\left(i\hbar\frac{\partial}{\partial t}\right)^2 \Psi = i\hbar\frac{\partial}{\partial t}(H\Psi) = H\left(i\hbar\frac{\partial \Psi}{\partial t}\right) = H^2\Psi,$$

provided H is not explicitly dependent on t. And so on. So

$$\Psi(x, t+t_0) = \sum_{n=0}^{\infty} \frac{1}{n!} t_0^n \left(-\frac{i}{\hbar}H\right)^n \Psi = e^{-iHt_0/\hbar}\Psi(x,t).$$

(c)

$$\langle Q\rangle_{t+t_0} = \langle \Psi(x,t+t_0)|Q(x,p,t+t_0)|\Psi(x,t+t_0)\rangle.$$

But $\Psi(x,t+t_0) = e^{-iHt_0/\hbar}\Psi(x,t)$, so, using the hermiticity of H to write $\left(e^{-iHt_0/\hbar}\right)^\dagger = e^{iHt_0/\hbar}$:

$$\langle Q\rangle_{t+t_0} = \langle \Psi(x,t)|e^{iHt_0/\hbar} Q(x,p,t+t_0) e^{-iHt_0/\hbar}|\Psi(x,t)\rangle.$$

If $t_0 = dt$ is very small, expanding to first order, we have:

$$\langle Q\rangle_t + \frac{d\langle Q\rangle}{dt}dt = \langle\Psi(x,t)|\underbrace{\left(1+\frac{iH}{\hbar}dt\right)\left[Q(x,p,t)+\frac{\partial Q}{\partial t}dt\right]\left(1-\frac{iH}{\hbar}dt\right)}_{\star}|\Psi(x,t)\rangle$$

$$\left[\star = Q(x,p,t) + \frac{iH}{\hbar}dt\,Q - Q\left(\frac{iH}{\hbar}dt\right) + \frac{\partial Q}{\partial t}dt = Q + \frac{i}{\hbar}[H,Q]dt + \frac{\partial Q}{\partial t}dt\right]$$

$$= \langle Q\rangle_t + \frac{i}{\hbar}\langle[H,Q]\rangle dt + \langle\frac{\partial Q}{\partial t}\rangle dt.$$

$$\therefore \frac{d\langle Q\rangle}{dt} = \frac{i}{\hbar}\langle[H,Q]\rangle + \langle\frac{\partial Q}{\partial t}\rangle. \quad \text{QED}$$