PHYS401 Quantum Physics I - FINAL EXAM Spring 2012

No books, calculators, or notes Name: Solutions

1. Consider an electron bound to a two-dimensional infinite quantum well with sides of length $x = \sqrt{8}L$ and $y = \sqrt{3}L$.
   a. Write down the time-independent differential wave equation governing the energy of this system. (1)
   b. Solve this equation for the stationary-state wavefunctions $\Psi(x, y)$, and determine all the allowed energies, using quantum numbers n_x and n_y. What is the lowest "ground-state" energy?(2)
   c. Calculate the energies for the next three higher energy levels. For each distinct energy value, list the possible combinations of n_x and n_y and the degree of degeneracy. (3)

(a) $$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\psi(x,y) = E\,\psi(x,y) \quad \text{for} \quad \begin{matrix} 0 < x < \sqrt{8}L \\ 0 < y < \sqrt{3}L \end{matrix}$$

energy eigenvalue

(b) By separation of variables, $\psi(x,y) = \psi_x(x)\psi_y(y)$ leads to two ordinary differential eqns

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_x(x) = E_x\,\psi_x(x)$$

$$-\frac{\hbar^2}{2m}\frac{d^2}{dy^2}\psi_y(y) = E_y\,\psi_y(y) \qquad \text{where } E = E_x + E_y$$

applying B.C.'s $\psi_x(x=0,\sqrt{8}L)=0$ and $\psi_y(x, y=0,\sqrt{3}L)=0$ gives eigenvalues s.t.

$$E = \frac{\hbar^2}{2m}\left(\frac{\pi^2 n_x^2}{8L^2} + \frac{\pi^2 n_y^2}{3L^2}\right) \quad \text{and}$$

$$\psi(x,y) = A\sin(k_{n_x}x)\sin(k_{n_y}y) \qquad \text{where } A = \sqrt{\frac{2}{\sqrt{8}L}}\sqrt{\frac{2}{\sqrt{3}L}}$$

Ground state ($n_x = n_y = 1$) has energy $E = \frac{\hbar^2\pi^2}{2mL^2}\left(\frac{1}{8}+\frac{1}{3}\right) = \frac{\hbar^2\pi^2}{2mL^2}\frac{11}{24}$

(c)

n_x	n_y	$E\left(\frac{\hbar^2\pi^2}{2mL^2}\right)$	degeneracy
2	1	$\frac{4}{8}+\frac{1}{3} = \frac{20}{24}$	1
1	2	$\frac{1}{8}+\frac{4}{3} = 35/24$	2
3	1	$\frac{9}{8}+\frac{1}{3} = 35/24$	
2	2	$\frac{4}{8}+\frac{4}{3} = 44/24$	1

2. The angular part of the wavefunction for an electron bound in a hydrogen atom is $\psi(\theta,\phi) = C(5Y_4^3 + Y_6^3 + Y_6^0)$, where $Y_l^m(\theta,\phi)$ are the normalized spherical harmonics.
   a. What is the value of normalization constant C?(1)
   b. What is the probability of finding the atom in a state with m=3?(2)
   c. What are the expectation values of angular momentum operators L_z and L^2?(4)

(a) $\int \psi^* \psi \, d\Omega = 1$. Since Y_ℓ^m is normalized, $C^2 \cdot (5^2 + 1^2 + 1^2) = 1$

So $C = \frac{1}{\sqrt{27}}$

(b)
$$\frac{5^2 + 1^2}{27} = \frac{26}{27}$$

(c) Y_ℓ^m are eigenstates of L_z and L^2 with eigenvalues $\hbar m$ and $\hbar^2 \ell(\ell+1)$,

Therefore,
$$\langle L_z \rangle = \frac{25}{27}\hbar \cdot 3 + \frac{1}{27}\hbar \cdot 3 + \frac{1}{27}\hbar \cdot 0 = \frac{26}{27} \cdot 3\hbar$$

$$\langle L^2 \rangle = \frac{25}{27}\hbar^2 \cdot 4(4+1) + \frac{1}{27}\hbar^2 6(6+1) + \frac{1}{27}\hbar^2 6(6+1)$$

$$= \frac{\hbar^2}{27}(25 \cdot 20 + 42 + 42) = \frac{584}{27}\hbar^2$$

3. The wavefunction $\psi(x) = Ae^{-b^2x^2}$ is the ground state of a one-dimensional harmonic oscillator. *A* and *b* are real constants.
    a. What are the units of *A* and *b*?(2)
    b. Normalize the wavefunction to determine the value of *A* (assume it is real).(2)
    c. What is the potential *V(x)* in terms of *ħ*, *m*, and *b*?(2)
    d. What is the energy of this state? What is the energy of the first excited state? (2)

(a) Since $\int \psi^*\psi\, dx = 1$, A must be units of length$^{-1/2}$.
Since b^2x^2 is unitless, b has units of length^{-1}

(b) $\int \psi^*\psi\, dx = A^2\int e^{-2b^2x^2} dx = A^2\sqrt{\frac{\pi}{2b^2}} = 1$ so $A = \left(\frac{2}{\pi}\right)^{1/4} b^{1/2}$

(c) for $V(x) = \frac{1}{2}m\omega^2x^2$, ground state wavefunction is $\propto e^{-\frac{m\omega}{2\hbar}x^2}$
Therefore, $b^2 = \frac{m\omega}{2\hbar}$ and $\omega = \frac{2\hbar b^2}{m}$ so that
$V(x) = \frac{1}{2}m\left(\frac{2\hbar b^2}{m}\right)^2 x^2 = \frac{2}{m}\hbar^2 b^4 x^2$

(d) $E = \hbar\omega(n+\frac{1}{2})$, $n = 0, 1, 2\ldots$ so ground state has $E = \frac{\hbar\omega}{2} = \frac{\hbar^2 b^2}{m}$
1st excited state $(n=1)$ has $E = \frac{3\hbar\omega}{2} = 3\frac{\hbar^2 b^2}{m}$

4. Protons have spin ½, just like electrons. However, their mass is approximately 1 GeV/c^2, nearly 2000 times larger. Decide whether their associated magnetic moment is (greater than/less than/ equal to) the electron magnetic moment μ_B, and explain why. (3)

Magnetic moment is

$$\vec{\mu} = \text{current} \cdot \text{area} = -ef \cdot \pi r^2 = e \frac{v}{2\pi r} \pi r^2 = -\frac{evr}{2} = \frac{e(mvr)}{2m} = \left(\frac{e\hbar}{2m}\right)\frac{\vec{L}}{\hbar}$$

$$= \mu_B \frac{\vec{L}}{\hbar}$$

Since $\mu_B \propto \frac{1}{m}$, the proton magnetic moment will be $\sim \times 2000$ smaller!

5. Consider a three-level system where the Hamiltonian and observable ***A*** are given by the matrix operators $\hat{H} = \hbar\omega \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ and $\hat{A} = \mu \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.
   a. What are the possible values obtained in a measurement of A? (2)
   b. Does a state exist in which both the results of a measurement of energy *E* and observable *A* can be predicted with certainty? Why or why not? (2)
   c. Two measurements of A are carried out, separated in time by *t*. If the result of the first measurement is its largest possible value, determine the expectation value $\langle\psi(t)|A|\psi(t)\rangle$ for the second measurement. (3)

(a) eigenvalues of $\hat{A}$ are roots of characteristic eqn

$$\det\begin{vmatrix} -\lambda & \mu & 0 \\ \mu & -\lambda & \mu \\ 0 & \mu & -\lambda \end{vmatrix} = 0 \rightarrow -\lambda\left(\lambda^2-\mu^2\right)+\lambda\mu^2=0 \Rightarrow \lambda = 0, \pm\sqrt{2}\mu$$

(b) NO, because $\hat{H}$ and $\hat{A}$ do not have the same eigenvectors

(c) eigenvector associated with $\lambda = +\sqrt{2}\mu$ is

$$\mu\begin{bmatrix} -\sqrt{2} & 1 & 0 \\ 1 & -\sqrt{2} & 1 \\ 0 & 1 & -\sqrt{2} \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{aligned} -\sqrt{2}a+b&=0 \\ a-\sqrt{2}b+c&=0 \\ b-\sqrt{2}c&=0 \end{aligned} \Rightarrow \begin{aligned} a&=1/2 \\ b&=\sqrt{2}/2 \\ c&=1/2 \end{aligned}$$

Now decompose into eigenvectors of $\hat{H}$ and add time-dependent phase:

$$\psi(t) = \frac{1}{2}\left[\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{-i\omega t} + \begin{bmatrix} 0 \\ \sqrt{2} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{+i\omega t}\right] = \frac{1}{2}\begin{bmatrix} e^{-i\omega t} \\ \sqrt{2} \\ e^{+i\omega t} \end{bmatrix}$$

Then, $\langle\psi(t)|\hat{A}|\psi(t)\rangle =$

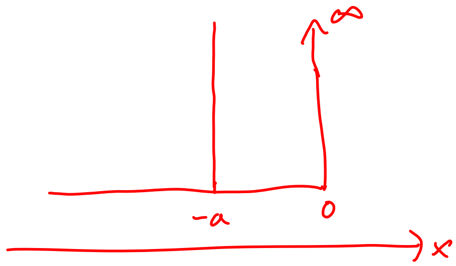
$$\frac{1}{2}\begin{bmatrix} e^{+i\omega t} & \sqrt{2} & e^{-i\omega t} \end{bmatrix} \mu\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \frac{1}{2}\begin{bmatrix} e^{-i\omega t} \\ \sqrt{2} \\ e^{+i\omega t} \end{bmatrix} = \frac{\mu}{4}\begin{bmatrix} e^{i\omega t} & \sqrt{2} & e^{-i\omega t} \end{bmatrix}\begin{bmatrix} \sqrt{2} \\ 2\cos\omega t \\ \sqrt{2} \end{bmatrix} = \sqrt{2}\mu\cos\omega t$$

6. Consider an electron incident from $x = -\infty$ on the following one-dimensional potential:

$$V(x) = \begin{cases} D\delta(x+a), x < 0 \\ \infty, x \geq 0 \end{cases}$$

a. Draw the potential. (1)
b. What are the units of D? (1)
c. What are the boundary conditions on the wavefunction at $x=-a$? (2)
d. What is the absolute magnitude of the reflection coefficient, and why? (2)

a



b $\left[\int V(x)\,dx\right]$ = length · energy so

$\int D\,\delta(x+a)\,dx = D$ has same units.

c B.C.s given by $\displaystyle\int_{-a-\epsilon}^{-a+\epsilon}\left[-\frac{\hbar^2}{2m}\psi'' + D\delta(x+a)\psi = E\psi\right]dx$ (and ψ continuous)

$$-\frac{\hbar^2}{2m}\psi'\Big|_{-a-\epsilon}^{-a+\epsilon} + D\psi(-a) = 0 \Rightarrow \Delta\psi' = \frac{2mD}{\hbar^2}\psi(-a)$$

d $|r| = 1$ because $\psi = 0$ for $x > 0$ so no transmitted probability flux.

7. Consider the following angular part of the wavefunction for an electron in a hydrogen atom: $\psi(\theta,\phi) = \frac{1}{\sqrt{3}}\left(Y_1^0|\uparrow\rangle + \sqrt{2}Y_1^1|\downarrow\rangle\right)$ where $|\uparrow\rangle$ and $|\downarrow\rangle$ are eigenstates of s_z.
   a. Without any knowledge of the radial part of the wavefunction, what is the *minimum* value that could be returned as the result of an energy measurement? (1)
   b. With what probability will the result of a measurement of spin along z give $+\hbar/2$? (1)
   c. The total angular momentum is the sum of orbital and spin: *J=L+S*. Compute the expectation value of the operator $<J_z>$. (3)

(a) Since $l=1$, the minimum value of n is 2.

Therefore $E = \frac{-R_y}{n^2} = -\frac{R_y}{4} \sim \frac{-13.6eV}{4} \sim -3.4\,eV$

(b)
$$\frac{1^2}{(\sqrt{3})^2} = \frac{1}{3}$$

(c) Hilbert Space is $L \oplus S$ (6×6), $J_z = L_z \otimes \mathbb{1} + \mathbb{1} \otimes S_z$

$$\psi = \frac{1}{\sqrt{3}}|l=1, m=0\rangle \otimes |\uparrow\rangle + \frac{\sqrt{2}}{\sqrt{3}}|l=1, m=1\rangle \otimes |\downarrow\rangle = \begin{bmatrix} 0 \\ \sqrt{2/3} \\ 1/\sqrt{3} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

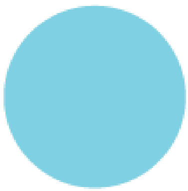
So $\langle\psi|J_z|\psi\rangle =$

$$\frac{1}{\sqrt{3}}\begin{bmatrix} 0 & \sqrt{2} & 1 & 0 & 0 & 0 \end{bmatrix} \hbar \begin{bmatrix} 3/2 & & & & & \\ & 1/2 & & & 0 & \\ & & \frac{1}{2} & & & \\ & 0 & & -\frac{1}{2} & & \\ & & & & -\frac{1}{2} & \\ & & & & & -3/2 \end{bmatrix} \frac{1}{\sqrt{3}}\begin{bmatrix} 0 \\ \sqrt{2} \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \frac{\hbar}{3}\begin{bmatrix} 0 & \sqrt{2} & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{\hbar}{3}\cdot\frac{3}{2} = \frac{\hbar}{2}$$

8. Identify the quantum numbers (*l* and *m*) for the following spherical harmonics found on the www: (4 total)

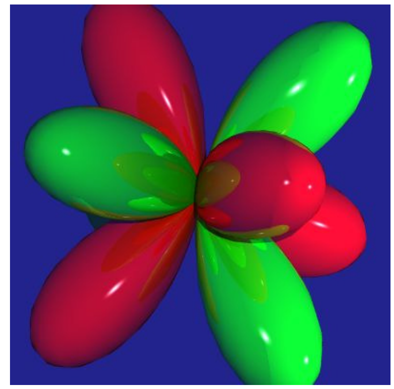
$l=0$
$m=0$



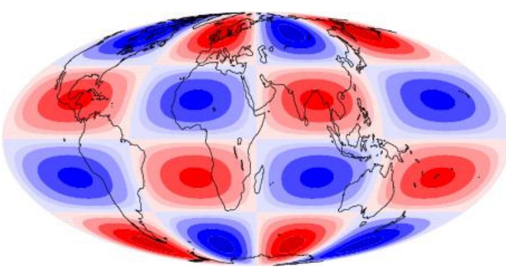
$l=1, m=0$

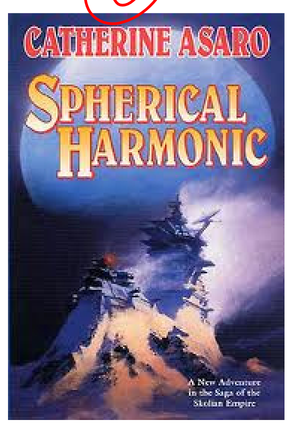
$l=2, m=0$

$l=3, m=\pm 2$

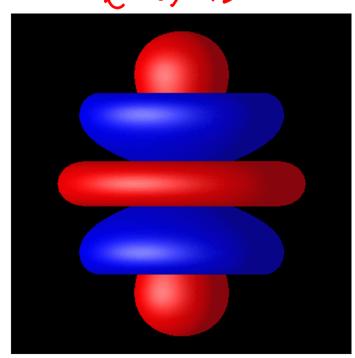


$l=5$
$m=\pm 2$

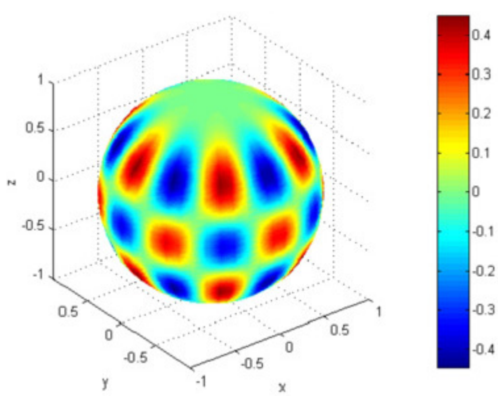




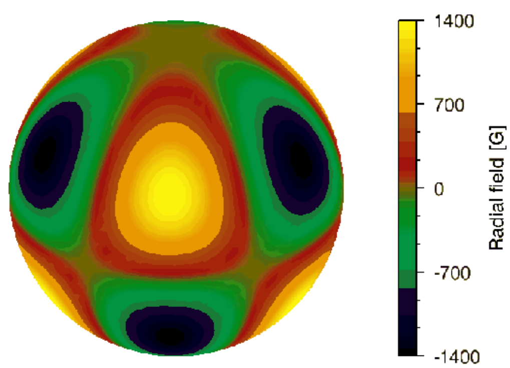
$l=3, m=0$



$l=8$
$m=\pm 6$



$l=4$
$m=\pm 3$



Potentially useful formulae:

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right)$$

$$\hat{p} = -i\sqrt{\frac{m\omega\hbar}{2}}\left(\hat{a} - \hat{a}^+\right)$$

$$[\hat{a}, \hat{a}^+] = 1$$

$$|n\rangle = \frac{1}{\sqrt{n!}}\left(\hat{a}^+\right)^n|0\rangle$$

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\frac{1}{\sqrt{2^n n!}}H_n(\xi)\exp\left(-\tfrac{1}{2}\xi^2\right), \quad \text{where } \xi = \sqrt{\frac{m\omega}{\hbar}}x$$

$$H_0 = 1 \qquad H_1 = 2\xi \qquad H_2 = 4\xi^2 - 2 \qquad H_2 = 8\xi^3 - 12\xi$$

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right)$$

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

$$\sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$L^2 = \hbar^2\,\ell(\ell + 1)$, $L_z = m\,\hbar$.