33 pts

PHYS401 Quantum Physics I - Exam 3 Spring 2012

No books, calculators, or notes Name: Solution

1. The Hamiltonian for a two-level system is represented by the matrix $H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

   a. Find the energy eigenvalues and eigenvectors.

$(\hat{H} - E)\varphi = 0$

$\begin{vmatrix} -E & 1 \\ 1 & -E \end{vmatrix} = 0$

$E^2 - 1 = 0$

$E = \pm 1$ 1

$E_1 = 1$: $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow |\varphi_1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ 1

$E_2 = -1$: $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow |\varphi_2\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ 1

   b. If the system starts out in a state $|S(0)\rangle = A\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, what is A (assuming it is real-valued) and what is $|S(t)\rangle$, the ket as a function of time $t>0$?

$A = \frac{1}{\sqrt{2}}$ s.t. $\langle s|s\rangle = 1$ 1

Since $|s\rangle = |\varphi_1\rangle$, $|S(t)\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-i\frac{E_1}{\hbar}t}$ 1

   c. If the system starts out in a state $|S(0)\rangle = B\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, what is B (assuming it is real-valued) and what is $|S(t)\rangle$? ~~What is the expectation value of the (hermitian) operator~~ $\cancel{G = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}}$ ~~as a function of time?~~

$B = 1$ s.t. $\langle s|s\rangle = 1$ 1

$|S(0)\rangle = \frac{1}{\sqrt{2}}(|\varphi_1\rangle - |\varphi_2\rangle)$ 1 so

$|S(t)\rangle = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-iE_1/\hbar\, t} - \frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-iE_2/\hbar\, t} = \begin{bmatrix} -i\sin\omega t \\ \cos\omega t \end{bmatrix}$ 1 $(\omega \equiv E_1/\hbar)$

(Since $E_2 = -E_1$)

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d. Calculate the four matrix elements of H in the basis $|1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} i \\ 1 \end{bmatrix}$, $|2\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} i \\ -1 \end{bmatrix}$

$$\langle 1|H|1\rangle = \frac{1}{2}\begin{bmatrix} -i & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} i \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -i & 1 \end{bmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} = 0$$

$$\langle 2|H|2\rangle = \frac{1}{2}\begin{bmatrix} -i & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} i \\ -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -i & -1 \end{bmatrix}\begin{bmatrix} -1 \\ i \end{bmatrix} = 0$$

$$\langle 1|H|2\rangle = \frac{1}{2}\begin{bmatrix} -i & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} i \\ -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -i & 1 \end{bmatrix}\begin{bmatrix} -1 \\ i \end{bmatrix} = i$$

$\langle 2|H|1\rangle = -i$ by symmetry

So

H in this basis is $\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$ 2

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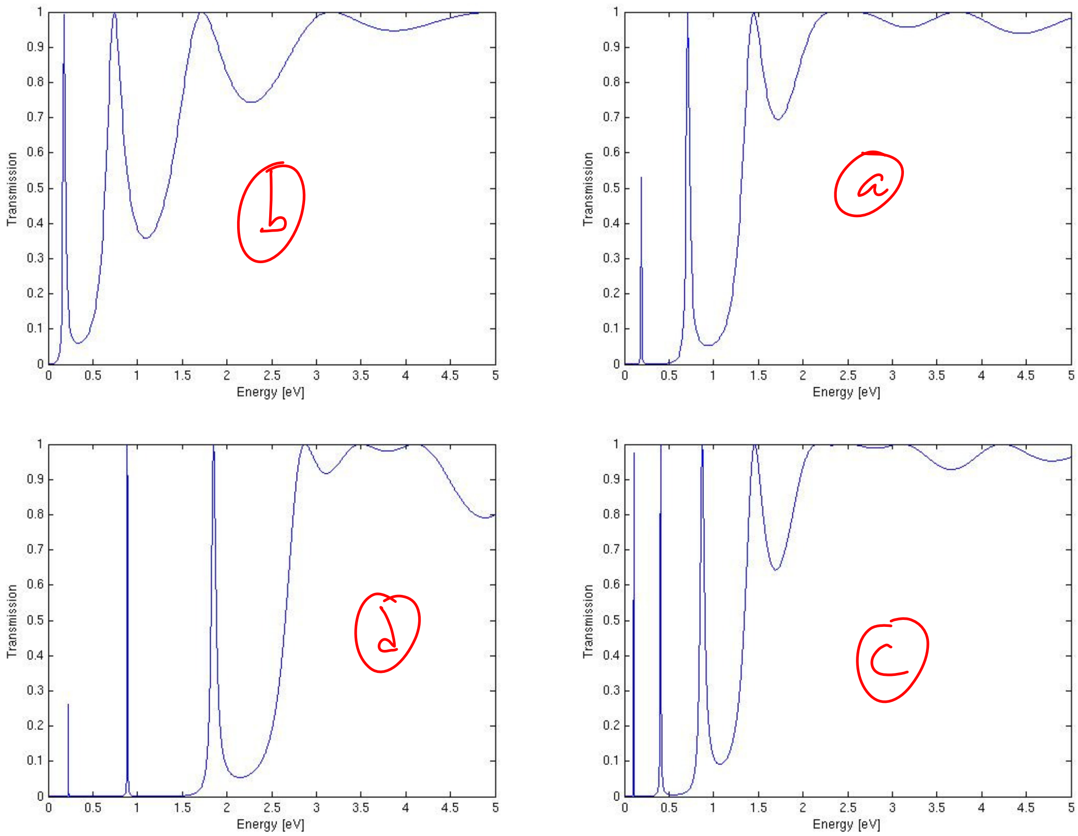
2. The following are transmission functions calculated for scattering through double barrier potentials. The sets of parameters used to define the potentials are

	Barrier width [nm]	Well width [nm]	Barrier height [eV]
a	0.5	1	1
b	0.25	1	1
c	0.5	1.5	1
d	0.5	1	2

By considering the effects on *resonance* features, match the parameter sets (a,b,c, or d) to the appropriate result below.



1 each

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3. An electron is incident from $x<0$ on a step-function scattering potential $V(x)=V_0\Theta(x)$.

a) Write down the propagating solutions of the 1-d time-independent Schrodinger Eq

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi + V\Psi = E\Psi$$

for both $x<0$ and $x>0$. Assume only a forward-propagating solution for $x>0$.

$x<0$: $\psi_- = e^{ik_1 x} + re^{-ik_1 x}$ $\quad \left(k_1 = \sqrt{\frac{2mE}{\hbar^2}}\right)$ 1

$x>0$: $\psi_+ = te^{ik_2 x}$ $\quad \left(k_2 = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}\right)$ 1

b) What are the boundary conditions at $x=0$?

$\psi_-(0) = \psi_+(0)$ 1

$\psi_-'(0) = \psi_+'(0)$ 1

c) Apply these boundary conditions and determine the amplitude of the transmitted wavefunction.

$1 + r = t \rightarrow r = t-1$ 1

$ik_1 - ik_1 r = ik_2 t$ 1

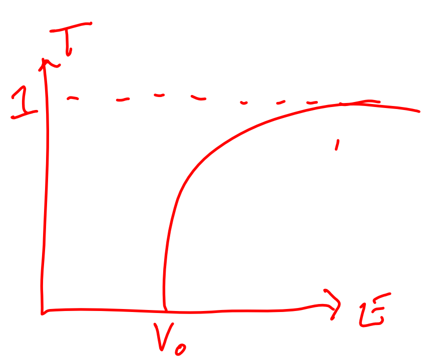
$k_1(1-(t-1)) = k_2 t$

$2k_1 = (k_1+k_2)t \rightarrow t = \frac{2k_1}{k_1+k_2}$ 1

d) What is the transmission coefficient T (the ratio of transmitted probability current to incident probability current)? [hint: probability current is the product of wavefunction group velocity and particle density] Plot T(E), and label axes and important values.

$$T = \frac{J_{trans}}{J_{inc}} = \frac{\frac{\hbar k_2}{m}\cdot|t|^2}{\frac{\hbar k_1}{m}\cdot 1} = \frac{k_2}{k_1}\frac{4k_1^2}{(k_1+k_2)^2} = \frac{4k_1k_2}{(k_1+k_2)^2}$$ 1 1

- As $E\to\infty$, $k_2\to k_1$ so $T\to 1$
- For $E=V_0$, $k_2=0$ so $T=0$



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4. An electron is bound in a 3-dimensional isotropic harmonic oscillator potential $V(\vec{r}) = \frac{1}{2}m\omega^2|\vec{r}|^2$, where $\vec{r} = \sqrt{x^2+y^2+z^2}$.

a. Write down the corresponding time-independent Schrodinger equation in Cartesian (x,y,z) coordinates.

$$\left[-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)+\frac{1}{2}m\omega^2\left(x^2+y^2+z^2\right)\right]\psi = E\psi$$

1

b. Use separation of variables to convert this partial differential equation into three ordinary differential equations.

$$\psi = \psi_{xy}\psi_z$$

$$\frac{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2}\psi_z+\frac{1}{2}m\omega^2 z^2\psi_z}{\psi_z} - E = \frac{\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\psi_{xy}+\frac{1}{2}m\omega^2(x^2+y^2)\psi_{xy}}{\psi_{xy}} = -E_{xy}$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2}\psi_z+\frac{1}{2}m\omega^2 z^2\psi_z = (E-E_{xy})\psi_z \quad 1 \quad \Rightarrow \quad E-E_{xy}=E_z=\hbar\omega\left(n_z+\tfrac{1}{2}\right)$$

$n_z = 0,1,2\ldots$

$$\psi_{xy}=\psi_x\psi_y$$

$$\frac{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi_x+\frac{1}{2}m\omega^2 x^2\psi_x}{\psi_x} - E_{xy} = \frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2}\psi_y-\frac{1}{2}m\omega^2 y^2\psi_y = -E_y$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi_x+\frac{1}{2}m\omega^2x^2\psi_x = (E_{xy}-E_y)\psi_x \quad 1 \quad \Rightarrow E_{xy}-E_y=E_x=\hbar\omega\left(n_x+\tfrac{1}{2}\right)$$

$n_x = 0,1,2\ldots$

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2}\psi_y+\frac{1}{2}m\omega^2y^2\psi_y = E_y\psi_y \quad 1 \quad \Rightarrow \quad E_y=\hbar\omega\left(n_y+\tfrac{1}{2}\right)$$

$n_y = 0,1,2\ldots$

Note That $(E-E_{xy})+(E_{xy}-E_y)+E_y = E_z+E_x+E_y = E!$ 1

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c. What are the possible total energy eigenvalues? What are the degeneracies of the first three energy levels?

$$E = E_x + E_y + E_z = \hbar\omega\left[\left(n_x+\tfrac{1}{2}\right)+\left(n_y+\tfrac{1}{2}\right)+\left(n_z+\tfrac{1}{2}\right)\right] = \hbar\omega\left(n_x+n_y+n_z+\tfrac{3}{2}\right)$$

$n_x, n_y, n_z = 0, 1, 2, \ldots$ 1

$E_{n_x n_y n_z}$	degeneracy
$E_{000} \rightarrow$	1
$E_{100} \rightarrow$	3
$E_{110/200} \rightarrow$	6

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d. What shape do the constant electron density surfaces form in the ground state?

$$\psi^*_{000}\psi_{000} \sim e^{-\frac{x^2}{2\Delta x^2}} e^{-\frac{y^2}{2\Delta y^2}} e^{-\frac{z^2}{2\Delta z^2}}$$

Since the potential is isotropic, $\Delta x = \Delta y = \Delta z$

and

$$\psi^*\psi \sim e^{-\frac{(x^2+y^2+z^2)}{2\Delta^2}} = e^{-\frac{r^2}{2\Delta^2}}$$ 1

This is constant at fixed radius $\rightarrow$ sphere

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