PHYS401 Quantum Physics I - Exam 2 Spring 2012

No books, calculators, or notes Name: Solutions

1. (2 pts each) Using **only** eV, cm, and s, what are the units of:

   a. Mass *m*

   $E = mc^2 \Rightarrow [m] = \frac{eV \cdot s^2}{cm^2}$

   b. Wavenumber *k*

   $e^{ikx} \Rightarrow [k] = cm^{-1}$

   c. Quantum number *n*

   e.g. $E = \hbar\omega(n + \frac{1}{2}) \Rightarrow [n] = 1$ (unitless)

   d. Reduced Planck constant $\hbar$

   $E = \hbar\omega \Rightarrow [\hbar] = eV \cdot s$

   e. Potential *V(x)*

   $-\frac{\hbar^2}{2m}\psi'' + V\psi = E\psi \Rightarrow [V(x)] = eV$

   f. Wavefunction $\Psi(x,t)$ [hint: consider normalization condition]

   $\int_{-\infty}^{\infty} \Psi^*\Psi \, dx = 1 \Rightarrow [\Psi^*\Psi] = cm^{-1}$

   So $[\Psi] = cm^{-1/2}$

2. An electron is in an equal superposition of the two lowest eigenstates of the simple harmonic oscillator potential $V(x) = \frac{1}{2}m\omega^2x^2$. We can therefore write the initial wavefunction as $\Psi(x,0) = A[\psi_0 + \psi_1]$, where ψ_n is the n^{th} normalized solution to the time-independent Schrodinger equation.

   a. (1 pt) Determine the value of A.

$$\int_{-\infty}^{\infty} \Psi^*\Psi dx = |A|^2 \int_{-\infty}^{\infty} (\psi_0^*\psi_0 + \psi_0^*\psi_1 + \psi_1^*\psi_0 + \psi_1^*\psi_1) dx$$

$$= 2|A|^2 = 1 \quad \text{So} \quad A = \frac{1}{\sqrt{2}}$$

   b. (2 pts) What is the expectation value of energy <E>?

$$H\psi_0 = E_0\psi_0 \quad \text{and} \quad H\psi_1 = E_1\psi_1 \quad \text{so}$$

$$\int_{-\infty}^{\infty} \Psi^* H \Psi dx = \frac{1}{2}\int_{-\infty}^{\infty} (\psi_0^* H\psi_0 + \psi_0^* H\psi_1 + \psi_1^* H\psi_0 + \psi_1^* H\psi_1) dx$$

$$= \frac{1}{2}(E_0 + E_1) = \frac{1}{2}\left(\hbar\omega/2 + \frac{3\hbar\omega}{2}\right) = \hbar\omega$$

   c. (2 pts) Write down the wavefunction $\Psi(x,t)$ for all t>0.

$$\Psi(x,t) = \frac{1}{\sqrt{2}}\left(\psi_0 e^{-i\frac{E_0}{\hbar}t} + \psi_1 e^{-i\frac{E_1}{\hbar}t}\right)$$

   d. (2 pt) What is the probability density for this wavefunction for all t>0?

$$\Psi^*\Psi = \frac{1}{2}\left(\psi_0^*\psi_0 + \psi_1^*\psi_1 + \psi_0^*\psi_1 e^{-i\frac{(E_1-E_0)}{\hbar}t} + \psi_1^*\psi_0 e^{+i\frac{(E_1-E_0)}{\hbar}t}\right)$$

$$= \frac{1}{2}\left(|\psi_0|^2 + |\psi_1|^2 + 2\psi_0\psi_1 \cos\omega t\right)$$

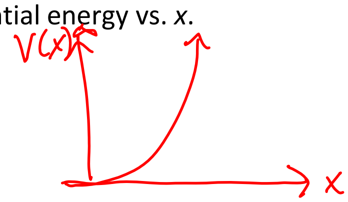
   e. (2 pts) Write an expression for <x> for all t>0 in terms of ψ_0 and ψ_1. Is this wavefunction a "stationary state"?

$$\langle x\rangle = \int_{-\infty}^{\infty} \Psi^* x \Psi dx = \frac{1}{2}\left(\int_{-\infty}^{\infty} \psi_0^* x\psi_0 dx + \int_{-\infty}^{\infty} \psi_1^* x\psi_1 dx + 2\cos\omega t \int_{-\infty}^{\infty} \psi_0^* x\psi_1 dx\right)$$

Since ψ_0 and ψ_1 have definite symmetry, $|\psi_0|^2$ and $|\psi_1|^2$ are even and the first two integrals are zero. The third term is time-dependent so this is <u>not</u> a stationary state (which is obvious since it is not an eigenstate)

3. Consider an electron with mass *m* in the potential $V(x) = \begin{cases} \frac{1}{2}kx^2, x > 0 \\ \infty, x < 0 \end{cases}$.

a. (1 pt) Draw the potential energy vs. *x*.



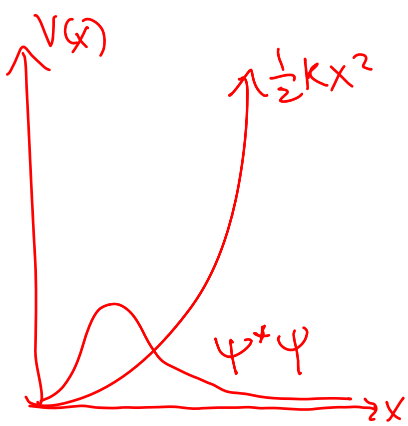
b. (1 pt) What is the boundary condition on the wavefunction at *x=0*?

$\Psi(x=0,t)=0$

c. (2 pts) In terms of *k*, *m*, and *ħ*, find the ground-state energy of the electron in this potential. [hint: what wavefunction solutions to the harmonic oscillator satisfy the boundary conditions?]

SHO Sol'n ψ_1 with energy $E_1 = \frac{3\hbar\omega}{2}$ has a node at $x=0$. Since the SHO potential $V(x) = \frac{1}{2}m\omega^2 x^2$, $K = m\omega^2$ so the ground state here is $E_0 = \frac{3}{2}\hbar\sqrt{\frac{K}{m}}$

d. (2 pts) Draw the probability density vs. *x* for this state superimposed on the potential.



4. Consider an electron of mass *m* in a 1-d infinite square potential $V(x) = \begin{cases} 0, 0 < x < a \\ \infty, otherwise \end{cases}$.

   a. (2 pts) Write down the time-independent Schrodinger equation for *0<x<a,* and determine the general solution.

$$-\frac{\hbar^2}{2m}\varphi'' = E\varphi \Rightarrow \varphi(x) = A\sin Kx + B\cos Kx,$$
$$\text{where } K = \sqrt{\frac{2mE}{\hbar^2}}$$

   b. (3 pts) What are the boundary conditions at *x=0 and x=a*? Apply them to the general solution to obtain the eigenstates and eigenenergies.

$\varphi(x=0) = 0$ and $\varphi(x=a) = 0$.

The first gives $B=0$ and the second gives

$K \to K_n = \frac{n\pi}{a}$ where $n = 1, 2, 3, \ldots$ "quantum number"

$\varphi_n = A\sin K_n x$

   c. (2 pts) Normalize the eigenstates.

$$\int_{-\infty}^{\infty} \varphi^*\varphi\, dx = \int_0^a |A|^2 \sin^2 K_n x\, dx = |A|^2 \int_0^a \left(\frac{1}{2} - \frac{1}{2}\cos 2K_n x\right) dx$$
$$= |A|^2 \frac{a}{2} = 1 \quad \text{so} \quad A = \sqrt{\frac{2}{a}}$$

   d. (1 pt) Suppose the electron is in the lowest-energy normalized eigenstate ("ground state"), when suddenly the well doubles in size so that V(x)=0 for *0<x<2a*. What is the new ground state eigenfunction?

$$\varphi_0^{new}(x) = \sqrt{\frac{2}{2a}} \sin\frac{\pi x}{2a} = \frac{1}{\sqrt{a}} \sin\frac{\pi x}{2a}$$

   e. (4 pts) What is the probability of now finding the electron in the *new* ground state of the wider well? [hint: sin(u)sin(v)=1/2(cos(u-v)-cos(u+v))]

$$C_0 = \int_0^a \frac{1}{\sqrt{a}} \sin\frac{\pi x}{2a} \sqrt{\frac{2}{a}} \sin\frac{\pi x}{a}\, dx = \frac{\sqrt{2}}{a} \int_0^a \frac{1}{2}\left(\cos\frac{\pi x}{2a} - \cos\frac{3\pi x}{2a}\right) dx$$
$$= \frac{1}{\sqrt{2}a}\left[\frac{2a}{\pi}\sin\frac{\pi x}{2a} - \frac{2a}{3\pi}\sin\frac{3\pi x}{2a}\right]\Bigg|_0^a = \frac{\sqrt{2}}{\pi}\left(1 - \left(-\frac{1}{3}\right) - 0\right) = \frac{4\sqrt{2}}{3\pi}$$

$$P_0 = |C_0|^2 = \frac{32}{9\pi^2}$$

5. A potential similar to the 1D infinite quantum well [with $V(x)=0$ for $-a<x<a$] has an attractive delta function potential in the center: $-\beta\delta(x)$.

   a. (2 pts) What are the units of β, and why?

   $\int \delta(x)\,dx = 1$ so $[\delta(x)] = cm^{-1}$,

   since $[-\beta\delta(x)] = eV$, $[\beta] = eV \cdot cm$

   b. (3 pts) By integrating the time-independent Schrodinger equation across $x=0$, derive the boundary condition on the first derivative of the wavefunction there.

$$\int_{-\epsilon}^{\epsilon}\left(-\frac{\hbar^2}{2m}\psi'' - \beta\delta(x)\psi\right)dx = \int_{-\epsilon}^{\epsilon} E\psi\,dx = 0$$

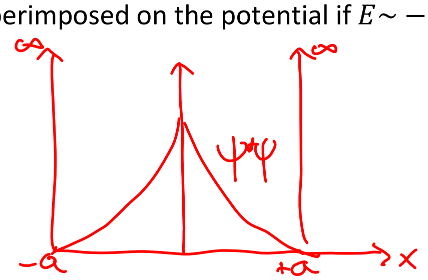
$$-\frac{\hbar^2}{2m}\left(\psi'(\epsilon) - \psi'(-\epsilon)\right) - \beta\psi(0) = 0$$

   c. (2 pts) Find the general solution to the Schrodinger equation for the bound state **with E<0** for the two regions $-a<x<0$ and $0<x<a$.

$$-\frac{\hbar^2}{2m}\psi'' = E\psi \Rightarrow \psi_+ = A_+e^{+Kx} + B_+e^{-Kx}$$

$$\psi_- = A_-e^{+Kx} + B_-e^{-Kx}, \quad K = \sqrt{-\frac{2mE}{\hbar^2}}$$

   d. (2 pts) Without solving the boundary condition equations, draw the probability density of this E<0 state, superimposed on the potential if $E \sim -\frac{\hbar^2}{2ma^2}$. Label axes etc.



   e. (1 pt) What is the expectation value <x> for this state?

   $\langle x \rangle = 0$ by symmetry