PHYS401 Quantum Physics I - Exam 1 Spring 2012

No books, calculators, or notes Name: Solution

1. Solve for the ***most general*** solution to the following linear, ordinary differential equations. (A, B, and C are fixed positive constants.) (1 pt each)

A. $\frac{d^2\Gamma}{dx^2} + C\Gamma = 0$, $\Gamma(x) = C_1 e^{+i\sqrt{C}x} + C_2 e^{-i\sqrt{C}x}$

$\Gamma'' = -C\Gamma$

undetermined coefs

B. $A\Psi = -\frac{d\Psi}{dy}$, $\Psi(y) = C_0 e^{-Ay}$

$\Psi' = -A\Psi$

C. $\frac{d^2\Theta}{dt^2} = 4$, $\Theta(t) = 2t^2 + C_1 t + C_2$

D. $\frac{d^2\Phi}{dr^2} \Big/ \Phi = B^2$, $\Phi(r) = C_1 e^{Br} + C_2 e^{-Br}$

$\Phi'' = B^2 \Phi$

2. The classical wave equation is

$$\frac{\partial^2}{\partial x^2} f(x,t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} f(x,t)$$

No boundary conditions:

a) Convert this partial differential equation of two variables into 2 ordinary differential equations: one for t and one for x. (3 pts)

$f(x,t) = X(x)\,T(t)$ then substitute:

$$\frac{X''T}{XT} = \frac{\frac{1}{c^2}X\ddot{T}}{XT}$$

undetermined constant

$$\frac{X''}{X} = \frac{\ddot{T}}{c^2 T} = -k^2 \longrightarrow \begin{cases} X'' = -k^2 X \\ \ddot{T} = -k^2 c^2 T \end{cases}$$

b) Find the most general solutions to both equations and write the full solution for $f(x,t)$, subject to a specific choice of initial conditions. (3 pts)

$$X(x) = \int_{-\infty}^{\infty} B(k)\, e^{ikx}\, dk$$

sums over all possible values of k

$$T(t) = \int_{-\infty}^{\infty} C(k)\, e^{ikct}\, dk$$

$$f(x,t) = \int_{-\infty}^{\infty} A(k)\, e^{ikx}\, e^{\pm ikct}\, dk$$

($A(k)$ accounts for undetermined $B(k)$ and $C(k)$)

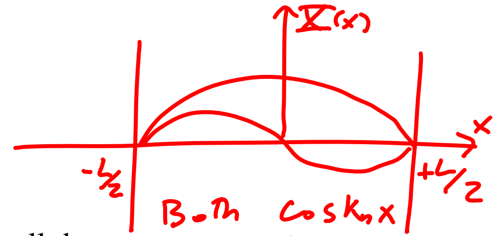
c) Show that your answer in part (b) is consistent with the "d'Alembert" solution $f(x,t) = f(x \pm ct)$. (1 pt)

$$f(x,t) = \int_{-\infty}^{\infty} A(k)\, e^{ik(x\pm ct)}\, dk$$

This is identical to the Fourier form

$$f(x) = \int_{-\infty}^{\infty} \tilde{f}(k)\, \frac{e^{ikx}}{\sqrt{2\pi}}\, dk \quad \text{except } x \to x \pm ct$$

So until BCs and ICs are specified, <u>any</u> function of $x \pm ct$, $f(x \pm ct)$, satisfies the wave eqn!



Both $\cos k_n x$ and $\sin k_n x$ satisfy BCs w/ correct k_n

3. Fixed boundary conditions:

a) If a string is held fixed at $f(x=-L/2,t)=f(x=+L/2,t)=0$, solve for all the eigenfunctions and discrete eigenvalues of the x-dependent equation from problem 2(a) for $-L/2<x<+L/2$. (3 pts)

It is appropriate to use the trig. form $X(x) = A\sin Kx + B\cos Kx$

Then, B.C.'s give

1) $x = L/2$: $A\sin\frac{KL}{2} + B\cos\frac{KL}{2} = 0$

2) $x=-L/2$: $A\sin\left(-\frac{KL}{2}\right) + B\cos\left(-\frac{KL}{2}\right) = -A\sin\frac{KL}{2} + B\cos\frac{KL}{2} = 0$

Adding: $2B\cos\frac{KL}{2} = 0$ so $K_n = \frac{2n+1}{L}\pi$ $n = 0, 1, 2, \ldots$ (2n+1 odd)

Subtracting: $2A\sin\frac{KL}{2} = 0$ so $K_n = \frac{2n}{L}\pi$ $n = 1, 2, \ldots$ (2n even)

b) If the initial conditions are $f(x,0)=\delta(x)$ and $df/dt|_{t=0}=0$, determine the full solution $f(x,t)$. (4 pts)

Since $\delta(x)$ is even, no odd contribution from $\sin k_n x$. Then,

$$\delta(x) = \sum_{n=0}^{\infty} B_n \cos\frac{2n+1}{L}\pi x$$

Project onto basis:

$$\int_{-L/2}^{L/2} \cos\frac{2m+1}{L}\pi x\, \delta(x)\,dx = \sum_{n=0}^{\infty} B_n \int_{-L/2}^{L/2} \cos\frac{2m+1}{L}\pi x \cos\frac{2n+1}{L}\pi x\,dx$$

$$1 = \sum_{n=0}^{\infty} B_n \delta_{nm}\cdot\left(\frac{L}{2}\right) = B_m L/2$$

So

$$f(x,t) = \sum_{n=0}^{\infty} \frac{2}{L}\cos K_n x \cos K_n c t$$

$$\left(K_n = \frac{2n+1}{L}\pi\right)$$

4. Periodic boundary conditions and finite differences

a) Using the "limit" definition of the derivative,

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\frac{\Delta x}{2}) - f(x-\frac{\Delta x}{2})}{\Delta x},$$

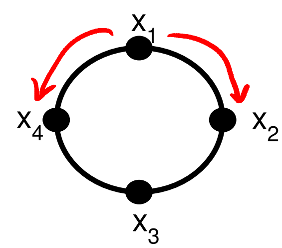
write the x-dependent differential equation from problem 2(a) in the finite-differences approximation at x_i.

$\frac{d^2 X}{dx^2} = -k^2 X$: second derivative $\frac{d^2 X}{dx^2} = \frac{d}{dx}\left(\frac{dX}{dx}\right)$ so

$$\frac{X'(x+\frac{\Delta x}{2}) - X'(x-\frac{\Delta x}{2})}{\Delta x} = \frac{\frac{X(x+\Delta x) - X(x)}{\Delta x} - \left(\frac{X(x) - X(x-\Delta x)}{\Delta x}\right)}{\Delta x} = \frac{X(x+\Delta x) - 2X(x) + X(x-\Delta x)}{\Delta x^2}$$

$$\Rightarrow \frac{X(x_{i+1}) - 2X(x_i) + X(x_{i-1})}{\Delta x^2} = -k^2 X(x_i)$$

b) Suppose a string is shaped in a loop. If we discretize the loop into just 4 segments at x_1, x_2, x_3, and x_4, each of length Δx, use your answer from 4(a) above to explicitly write down the four corresponding finite differences equations. NOTE: Since the boundary conditions are different, this problem is similar to, but **not** identical to the string between two fixed points discussed in class! *In particular, every segment is connected to two others (not true for the endpoints of the string with fixed B.C.s we treated in class)* (3 pts)



$i=1$: $-2X_1 + X_2 + X_4 = -k^2 \Delta x^2 X_1$

$i=2$: $X_1 - 2X_2 + X_3 = -k^2 \Delta x^2 X_2$

$i=3$: $X_2 - 2X_3 + X_4 = -k^2 \Delta x^2 X_3$

$i=4$: $X_1 + X_3 - 2X_4 = -k^2 \Delta x^2 X_4$

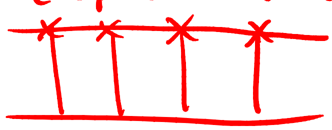
c) Write the four equations from 4(b) in matrix equation form $\boldsymbol{HX}=\lambda\boldsymbol{X}$, where $\boldsymbol{X}$ is the eigenvector of matrix $\boldsymbol{H}$, and λ is the corresponding eigenvalue. Show that the matrix $\boldsymbol{H}$ is Hermitian. (2 pts)

real and Symmetric, so $H^{+}=H$

$$\begin{bmatrix} -2 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \overbrace{-k^2\Delta x^2}^{\lambda} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}$$

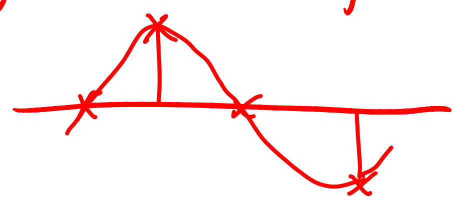
d) By analogy with the expected normal modes of the string, use your intuition to guess the form of the four eigenvectors of this matrix by plotting them. (2 pts)

1) Const. displacement $(k=0)$



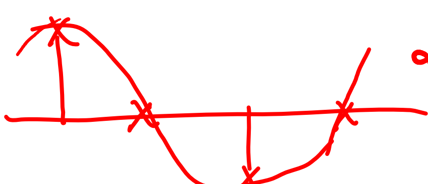
$\propto \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

3) one wavelength "sine"



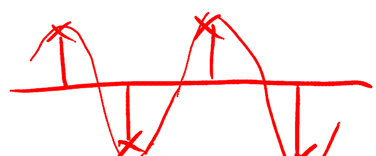
$\propto \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$

2) one wavelength "cosine"



$\propto \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$

4) two wavelengths



$\propto \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$

e) Show that the vector $\vec{X} = A\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$ is an eigenvector of H. What is its corresponding eigenvalue? (2 pts) Normalize $\vec{X}$ and determine the value of A. (1 pt)

$$\begin{bmatrix} -2 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 1 & -2 \end{bmatrix} A \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = A \begin{bmatrix} -4 \\ 4 \\ -4 \\ 4 \end{bmatrix} = -4 \cdot A \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$ so $\lambda = -4$

$|\vec{X}|^2 = 4A^2$ so $A = \frac{1}{2}$ normalizes $\vec{X}$ s.t. $|\vec{X}|^2 = 1$