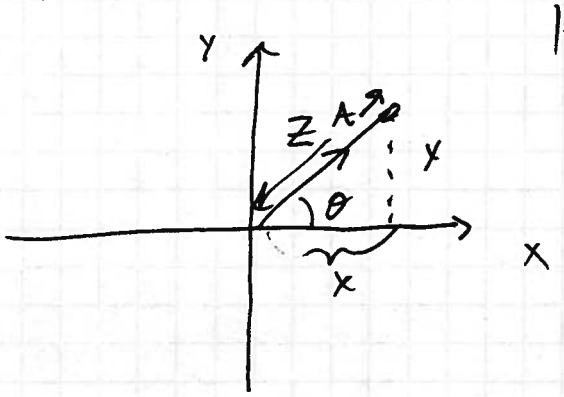
Complex numbers

$z = x + iy = Ae^{i\theta}$

$|z|^2 = z^* z$

$e^{i\theta} = \cos\theta + i\sin\theta$



$\theta = \tan^{-1}(y/x)$

$A = \sqrt{x^2 + y^2}$

$x = A\cos\theta$

$y = A\sin\theta$

Differential Eqs.

$\frac{dy(t)}{dt} = A \Rightarrow y(t) = At + C$

$\frac{dy(t)}{dt} = Ay(t) \Rightarrow y(t) = Ce^{At}$

$\frac{d^2y(t)}{dt} + Ay(t) = 0 \Rightarrow y(t) = C_1 e^{i\sqrt{A}t} + C_2 e^{-i\sqrt{A}t}$

Constants C_1, C_2 determined by initial conditions

Wave Dictionary

λ = wavelength

$k = \frac{2\pi}{\lambda}$ = wavenumber

T = period

ν = freq. $= \frac{1}{T}$

$\omega = 2\pi\nu = \frac{2\pi}{T}$

phase change due to $\Delta x = k(\Delta$

$\Delta t = \omega(\Delta t$

Planck formula for energy of a photon: $E = h\nu = \hbar\omega$

DeBroglie Wavelength $= \lambda = \frac{h}{p}$ ← particle momentum.

Thus $p = \frac{h}{\lambda} = \hbar k$

Expectation Value: $\langle C \rangle = \bar{C} = \sum_{i=1}^{N} P_i c_i$

(P_i: probability to observe c_i)

or $\langle C \rangle = \int_{\text{all possible } C_i} c\, P(c)\, dc$

Example: $\langle x \rangle = \int_{-\infty}^{\infty} x\, P(x)\, dx$

Variance: $\sigma^2 \equiv (\Delta C)^2 = \frac{1}{N} \sum_{i=1}^{N} (c_i - \langle C \rangle)^2$

$= \langle C^2 \rangle - \langle C \rangle^2$

Standard Dev: $\sigma = \Delta C = \sqrt{\langle C^2 \rangle - \langle C \rangle^2}$

Postulates of QM

I: For any observable A, we have $\hat{A} a = a a$

$\hat{A}$: an operator, a: an eigenvalue, a: an eigenfunction

Measurement of A always returns an eigenvalue (a).

II: The system is described by $\Psi(x,t)$.

Expectation values are calculated: $\langle C \rangle = \int \Psi^* \hat{C} \Psi\, dx$

$|\Psi|^2 dx$ is the probability to find the particle between x & $x + dx$.

III: Collapse of the Wavefunction. If A is measured, and e.v. (a) results, the wavefunction will be left in e.f. a_a.

IV: $\Psi(x,t)$ evolves in time as (when the system is not measured) $i\hbar \frac{\partial \Psi}{\partial t} = \hat{H} \Psi$, $\hat{H}$ = Hamiltonian = Energy operator

Common Operators	e.v. equation	Soln
$\hat{p} = -i\hbar \frac{d}{dx}$	$-i\hbar \frac{d}{dx} q(x) = p\, q(x)$	$q(x) = Ae^{ikx}$
$\hat{x} = x$	$\hat{x}\,\delta(x) = x\,\delta(x)$	$\delta(x)$ = Dirac Delta Function
$\hat{H} = \frac{\hat{p}^2}{2m} + V(x)$ $= -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$	$\hat{H} q = E q$	Soln depends on $V(x)$.

General Solution to the TDSE.

$$\Psi(x,t) = \sum_n a_n q_n(x) e^{-iE_n t/\hbar}$$ for discrete q_n.

a_n coeff. ↑ ↑ q_n e.f. of $\hat{H}$.

Determined by $\Psi(x,t=0) = \psi(x)$

Define $\omega_n \equiv E_n/\hbar$.

We call q_n the e.f. of $\hat{H}$ "stationary states" because all probabilities & expectation values are constant in time if $\Psi(x,t)$ is an e.f. of $\hat{H}$.

Particle in a Box: $V(x) = \begin{cases} 0, & 0 < x < L \\ +\infty, & \text{otherwise} \end{cases}$

soln: $q_n(x) = \sqrt{\frac{2}{L}} \sin\frac{n\pi x}{L}$, $E_n = \frac{\hbar^2 \pi^2 n^2}{2mL^2}$

For a particular initial state, $\{a_n\}$ can be calculated by

$$a_m = \int q_m^*(x)\, \psi(x)\, dx$$

$|a_m|^2$ can be interpreted as the probability to measure E_m, a particular energy e.v.

$$|a_m|^2 = P(E_m)$$

Particle-in-a-Box $\quad V(x) = \begin{cases} 0 & 0 < x < L \\ +\infty & \text{otherwise} \end{cases}$

$$\phi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \qquad E_n = \frac{\hbar^2 \pi^2 n^2}{2mL^2}$$

$\{\phi_n\}$ are ortho-normal:

$$\int \phi_m^* \phi_n \, dx = \delta_{mn} = \begin{cases} 1, & m = n \\ 0, & \text{otherwise} \end{cases}$$

$\{\phi_n\}$ are standing waves. Their form is determined by the boundary conditions.

Free-Particle

$$\hat{H} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \qquad \hat{H} \text{ \& } \hat{p} \text{ have common e.f.}$$

$\{\phi(x)\} = \{A e^{ikx}\}$ All e.f. of $\hat{p}$ are also e.f. of $\hat{H}$.

a continuum of e.f. & e.v.

$$E(k) = \frac{(\hbar k)^2}{2m}$$

$\phi_k(x)$ are not normalizable, but we can use them to construct normalizable states

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{ikx} \, dk \quad \text{for some } \phi(k).$$

Then $$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x)\, e^{-ikx}\, dx$$

$\phi(k)$ is the momentum space wavefunction.

$|\phi(k)|^2 dk$ = probability to observe momentum between k & $k+dk$.

General Soln: for free particle

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k)\, e^{+i(kx - \hbar k^2 t/2m)}\, dk$$

where $$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x)\, e^{-ikx}\, dx.$$

Quantum Mechanical Amplitudes - Complex numbers whose square is a probability.

Example: The $\{a_n\}$ coefficients: $|a_n|^2 = P(E_n)$

Example: The spatial wavefunction $\psi(x)$: $|\psi(x)|^2 = P(x)$

Example: The momentum wavefunction $\phi(k)$: $|\phi(k)|^2 = P(k)$

All three of these are equivalent ways of writing the same QM state. ⇒ If we know the $\{a_n\}$ for our system, then we also know $\psi(x)$: $\psi(x) = \sum_n a_n \phi_n(x)$

We can think of these amplitudes as the components of a vector: a state vector.

Ex: $\{a_n\}$ as a vector is $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \end{pmatrix}$ Column vector ↓ infinite # of components

The complex conjugate is a row vector

$\{a_n^*\}$ as a vector is $(a_1^*, a_2^*, a_3^*, a_4^*, \ldots)$ → row vector, infinite # of components

We write these vectors using Dirac notation:

$|$ a column vector $\rangle$ ← ket vector

$\langle$ a row vector $|$ ← row vector

a label to describe the vector.

Ex: In CM, $\vec{p}_{cat}$ = the momentum vector of a cat

(cat: a label)

In QM $|cat\rangle$ = the state vector of a cat

(cat: a label)

Ex: $|$ the $n=3$ eigenfunction of the particle-in-a-box $\rangle$

label

As a spatial wavefunction this state is $\sqrt{\frac{2}{L}}\sin\left(\frac{3\pi x}{L}\right)$.

As a column vector this state is

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \leftarrow a_3 = 1$$

everything else $= 0$.

As a row vector this is

$$(0, 0, \underset{\uparrow}{1}, 0, 0, \ldots\ldots)$$

$$a_3^* = 1$$

For discrete energy ~~eigenvalues~~ eigenfunctions we usually use "n" as a label:

$|n=3 \text{ eigenfunction}\rangle = |3\rangle$

"n=3 eigenfunction": a label; $|3\rangle$: a label, not a number

So we have a set of ket vectors $\{|n\rangle\}$

(n is a label, not an integer)

(take the dot product)

We can multiply a bra vector by a ket vector:

$$\langle \text{my QM state} | | \text{my QM state} \rangle$$

$$= (a_1^*, a_2^*, a_3^*, a_4^*, \ldots) \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \end{pmatrix}$$

$$= a_1^* a_1 + a_2^* a_2 + a_3^* a_3 + a_4^* a_4 + a_5^* a_5 + \ldots$$

$$= |a_1|^2 + |a_2|^2 + |a_3|^2 + \ldots$$

$$= \sum_n |a_n|^2$$ ← must sum to 1 since we are adding up all probabilities

$$= \underline{1}$$

We call a bra vector dotted with a ket vector a bracket:

Dirac $\langle$my QM state|my QM state$\rangle = 1$

↑ drop the double line.

↑ if my QM state is normalized.

In general a Dirac bracket between two different states is some complex number

$$\langle a|b\rangle = a_1^* b_1 + a_2^* b_2 + a_3^* b_3 + \ldots = \sum_n a_n^* b_n$$

↑ a label ↑ a label

= some complex number.

This is just like ordinary vector dot product:

$$\vec{a}\cdot\vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = \sum_{n=1}^{3} a_n b_n$$

What is this Dirac bracket:

$$\langle 3|\text{state } a\rangle = (0,0,1,0,0,0,\ldots)\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ \vdots \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \end{pmatrix}$$

↑ 3rd energy eigenstate

↑ $a_3 = 1$

$= b_3 a_3$

In general $\langle n | \text{state } a \rangle = a_n$ ← a complex #.

↑ the n^{th} energy expansion coefficient

We have "picked out" the n^{th} coefficient

This is just like ordinary vectors.

$$\vec{v} = v_x \hat{x} + v_y \hat{y}$$

$$\hat{x} \cdot \vec{v} = (1, 0)(v_x, v_y) = v_x$$

↑ dotting $\vec{v}$ with $\hat{x}$ picks out the x component

Let's label our arbitrary state as $|\psi\rangle$. Then

$\langle n|\psi\rangle = a_n$ ← a complex number

$|\langle n|\psi\rangle|^2 = P(E_n)$.

Ex: What is

$\langle n|m\rangle$?

↑ n^{th} energy eigenstate

↑ m^{th} energy eigenstate

Answer: $(\ldots, 0, 1, 0, \ldots)$

n^{th} position ↓

m^{th} position → 1

$= 0$ or 1

↑ if $n \neq m$

↑ if $n = m$

$\langle n|m\rangle = \delta_{nm}$

$\boxed{\langle n|m\rangle = \delta_{nm}}$ ← This is how we write orthonormality in Dirac Notation

Much cleaner than

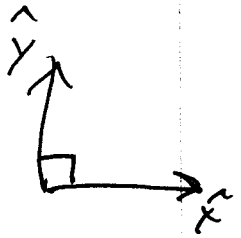
$$\int_{-\infty}^{\infty} \psi_n^*(x)\, \psi_m(x)\, dx = \delta_{nm}.$$

Compare to ordinary vectors:

Let $\hat{x}_1 = \hat{x}$ Then $\hat{x}_i \cdot \hat{x}_j = \delta_{ij}$

$\hat{x}_2 = \hat{y}$

$\hat{x}_3 = \hat{z}$.



$\hat{x}$ and $\hat{y}$ "have no overlap"

Similarly, $\langle n|$ and $|m\rangle$ "have no overlap"