The simplest bound-state system in QM is the particle-in-a-box, and its mathematical description is a Fourier Series.

Let $f(x)$ be ① periodic with period $2L$ and ② square integrable between $-L$ and L

Then $f(x)$ can be written as a "Fourier Series":

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L} \quad \text{for some set of coefficients } \{c_n\}}$$

!!!

This form can represent a real $f(x)$ or complex $f(x)$.

But if $f(x)$ is real then we can also write the Fourier Series as

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]}$$

!!!

for some set of $\{a_n\}$, $\{b_n\}$.

For real $f(x)$, we can convert between the two forms like this:

$$\boxed{c_n = \begin{cases} \frac{1}{2}(a_n - i b_n), & n > 0 \\ \frac{1}{2}(a_0), & n = 0 \\ \frac{1}{2}(a_{(-n)} + i b_{(-n)}), & n < 0 \end{cases}} \quad \text{AND} \quad \boxed{\begin{aligned} a_n &= c_n + c_{(-n)} \\ b_n &= i(c_n - c_{(-n)}) \end{aligned}}$$

To show that the two forms are equivalent, just substitute for c_n and use Euler's Formula:

$$e^{in\pi x/L} = \cos\left(\frac{n\pi x}{L}\right) + i\sin\left(\frac{n\pi x}{L}\right)$$

Two important properties of the set of function $\{e^{in\pi x/L}\}$:

① They are "complete": Any periodic, square integrable function can be written as a sum of the $\{e^{in\pi x/L}\}$.

② They are "orthonormal": If we multiply two of these functions together and integrate, we get:

Show this on the homework

$$\int_{-L}^{L}\Big(e^{i\underset{n!}{\underset{\uparrow}{n}}\pi x/L}\Big)\Big(e^{-i\underset{m!}{\underset{\uparrow}{m}}\pi x/L}\Big)\,dx = \begin{cases} 0, & \text{if } m \neq n \\ 2L, & \text{if } m = n \end{cases}$$

~~We write this~~

Define "Kronecker Delta": $\delta_{nm} \equiv \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases}$

Then we can write

$$\boxed{\frac{1}{2L}\int_{-L}^{L}\left(e^{in\pi x/L}\right)\left(e^{-im\pi x/L}\right)dx = \delta_{nm}}$$

↑ Orthonormality condition for the functions $\{e^{in\pi x/L}\}$.

Question: How do we find the correct $\{c_n\}$ ~~to~~ to represent a particular function $f(x)$?

Answer: Use "Fourier's Trick" method
↑ Griffith's terminology

Evaluate this integral:

$$\frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-im\pi x/L}\, dx = \frac{1}{2L}\int_{-L}^{L} \underbrace{\left(\sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}\right)}_{\text{Fourier series for } f(x)} e^{-im\pi x/L}\, dx$$

$$= \sum_{n=-\infty}^{\infty} c_n \underbrace{\frac{1}{2L}\int_{-L}^{L} \left(e^{in\pi x/L}\right)\left(e^{-im\pi x/L}\right) dx}_{\text{Kronecker Delta!}}$$

$$= \sum_{n=-\infty}^{\infty} c_n \delta_{nm} = c_m$$

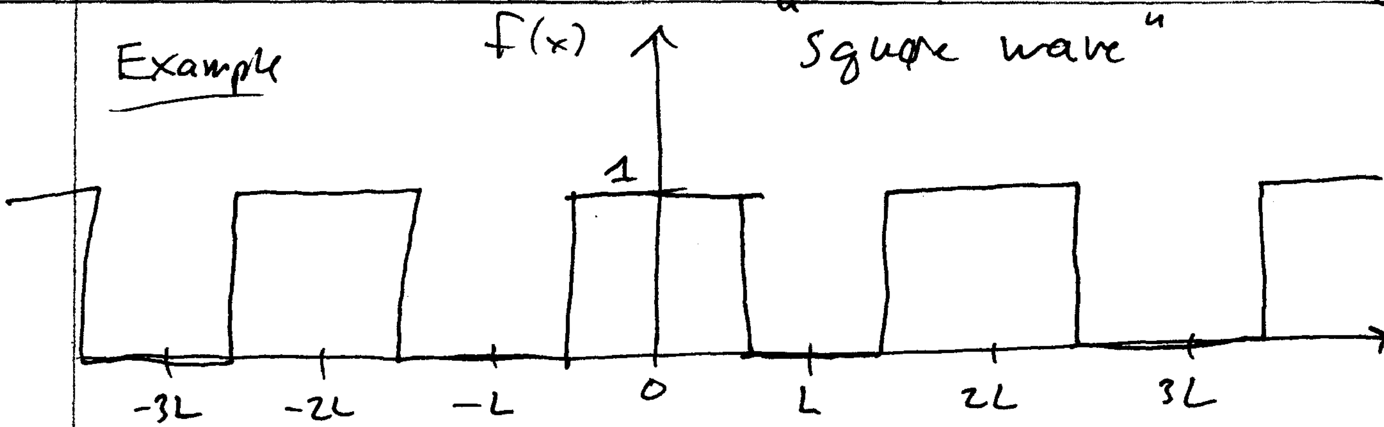
Kronecker Delta kills all terms except $n=m$

Conclusion To calculate a particular coefficient c_m, do this:

$$\boxed{c_m = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-im\pi x/L}\, dx}$$

one particular member of the set $\{c_n\}$.

Example



$$f(x) = \begin{cases} \emptyset, & -L < x < -\frac{L}{2} \\ 1, & -\frac{L}{2} < x < \frac{L}{2} \\ \emptyset, & \frac{L}{2} < x < L \end{cases}$$

Then $$c_m = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-im\pi x/L}\, dx$$

$$= \frac{1}{2L}\int_{-\frac{L}{2}}^{\frac{L}{2}} e^{-im\pi x/L}\, dx$$ ← f(x) only non-zero between $-\frac{L}{2}$ and $\frac{L}{2}$.

$$= \frac{1}{2L}\left(\frac{-L}{im\pi}\right)\left[e^{-im\pi x/L}\Big|_{-\frac{L}{2}}^{\frac{L}{2}} \right]$$

$$= \frac{-1}{2im\pi}\left[e^{-im\pi/2} - e^{+im\pi/2} \right]$$

$$\boxed{c_m = \frac{1}{m\pi}\sin\left(\frac{m\pi}{2}\right)}$$ ← true for $m \neq \emptyset$.

~~We can convert to the sine & cosine form:~~

$\boxed{\text{For } m = \emptyset}$ $$c_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{i(\emptyset)\pi x/L}\, dx = \frac{1}{2}$$

Convert to sine & cosine form:

$$a_n = c_n + c_{(-n)} = \frac{1}{n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{1}{(-n\pi)}\sin\left(-\frac{n\pi}{2}\right) = \boxed{\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)}$$

$$b_n = i\left(c_n - c_{(-n)}\right) \boxed{= \emptyset}$$ ← no sine terms!

f(x) is even, but sine terms are odd.

Summary: For the square wave,

$$f(x) = \sum_{\substack{n=-\infty \\ \text{except } n=0}}^{\infty} \left(\frac{1}{n\pi} \sin\left(\frac{n\pi}{2}\right)\right) e^{in\pi x/L} + \underbrace{\frac{1}{2}}_{n=0 \text{ term}}$$

AND ALSO

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right)\right) \cos\left(\frac{n\pi x}{L}\right)$$

Recap: Let $f(x)$ be ① periodic with period $2L$
and ② square integrable between $-L$ & L,

Then $f(x)$ can be written as a "Fourier Series":

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L} \quad \text{for some set of coefficients } \{c_n\}.}$$

This is very useful for periodic functions.
But what if we want to represent a non-periodic function?

In that case we can use the continuous limit of the Fourier series, called the "Fourier Transform".

Rewrite the Fourier Series this way:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L} \underbrace{\Delta n}_{1} = \frac{L}{\pi} \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L} \left(\frac{\pi \Delta n}{L}\right)$$

Define:

$$k \equiv \frac{n\pi}{L}, \quad \Delta k \equiv \frac{\pi \Delta n}{L}, \quad \text{and} \quad \frac{A(k)}{\sqrt{2\pi}} \equiv \frac{L}{\pi} c_n.$$

Then,

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} A(k) e^{ikx} \Delta k$$

Now, to represent a non-periodic function, take the limit where the period goes to infinity, $L \to \infty$:

Then $\Delta k = \frac{\pi \Delta n}{L} \to dk$ as $L \to \infty$

And $$\boxed{f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, A(k)\, e^{ikx}}$$

↑ Fourier representation of non-periodic $f(x)$.

The function $A(k)$ is called the "Fourier Transform" of $f(x)$.

$A(k)$ is analogous to the coefficients $\{c_n\}$, but the $\{c_n\}$ are for representing a periodic function, and $A(k)$ is for a non-periodic function.

Question

How do we find the correct $A(k)$ for a particular function $f(x)$?

Answer: For the c_n, we did this:

$$c_n = \frac{1}{2L} \int_{-L}^{L} f(x)\, e^{-in\pi x/L}\, dx$$

~~Take the limit where $L \to \infty$:~~ Rewrite: $\frac{n\pi}{L} = k$

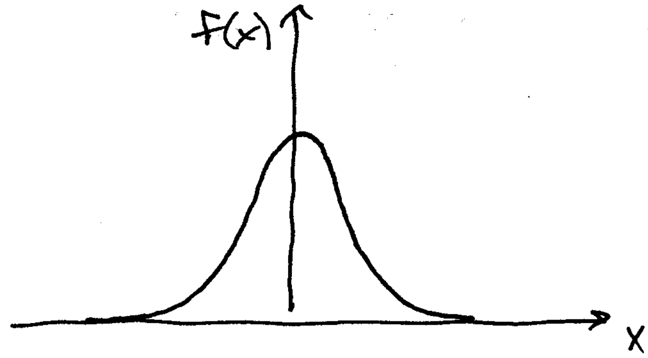
$$\underbrace{c_n \left(\frac{L}{\pi}\right) \sqrt{2\pi}}_{A(k)} = \frac{\sqrt{2\pi}}{2\pi} \int_{-L}^{L} f(x)\, e^{-ikx}\, dx$$

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-L}^{L} f(x)\, e^{-ikx}\, dx$$

Take the limit where $L \to \infty$:

$$\boxed{A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx} \Leftarrow \boxed{\text{Plancherel's Theorem}}$$

<u>Example</u>: Let $F(x) = e^{-ax^2}$ (a gaussian)



What is the Fourier Transform $A(k)$?

<u>Answer</u>:

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ax^2} e^{-ikx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(ax^2+ikx)}\, dx$$

Complete the square: Define $y \equiv \sqrt{a}\left(x + \frac{ik}{2a}\right)$.

Then $y^2 = ax^2 - \frac{k^2}{4a} + ikx$, $dy = \sqrt{a}\, dx$

$$\therefore \quad A(k) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{a}} e^{-k^2/4a} \int_{-\infty}^{\infty} e^{-(ax^2+ikx-\frac{k^2}{4a})} (\sqrt{a}\, dx)$$

$$= \frac{e^{-k^2/4a}}{\sqrt{2\pi a}} \underbrace{\int_{-\infty}^{\infty} e^{-y^2} dy}_{\sqrt{\pi}}$$

$$\boxed{A(k) = \frac{e^{-k^2/4a}}{\sqrt{2a}}}$$

← The Fourier Transform of a gaussian in x is a gaussian in k.