Hydrogen Wave functions

$$\psi(r,\theta,\phi) = R(r)\,Y_\ell^m(\theta,\phi)$$

$$R(r) = \frac{u(r)}{r}, \qquad u(r) = \rho^{\ell+1} e^{-\rho} \sum_{j=0}^{\infty} c_j \rho^j,$$

$$\rho \equiv \kappa r,$$

$$\kappa \equiv \frac{\sqrt{-2mE}}{\hbar}$$

Recursion relation for the coefficients $\{c_j\}$:

$$c_{j+1} = \left[\frac{2(j+\ell+1-n)}{(j+1)(j+2\ell+2)}\right] c_j$$

n = principle quantum number

for $Z=1$

Energy eigenvalues: $E_n = -\dfrac{mZ^2e^4k^2}{2\hbar^2}\,\dfrac{1}{n^2} = \dfrac{(-13.6\text{ eV})}{n^2}$

j is a non-negative integer, so for the power series to terminate, we must have

$$(j_{max} + \ell + 1 - n) = 0$$

$$\text{and} \quad \ell = (n-1) - \underbrace{j_{max}}_{0,1,2,3}$$

$$\therefore \quad \boxed{\ell \leq n-1}$$

$$\kappa = \frac{\sqrt{-2mE}}{\hbar} = \underbrace{\left(\frac{me^2 Zk}{\hbar^2}\right)}_{\equiv \frac{1}{a_0} = \text{"Bohr Radius"} = 0.529 \times 10^{-1}} \frac{1}{n} \equiv \frac{1}{a_0}\frac{1}{n}$$

a_0 units = length

What's the ground state wavefunction?

Ground state is $n=1,\ \ell=0,\ m=0.$

$$R(r) = \frac{1}{r}\rho^{\ell+1} e^{-\rho} \sum_{j=\emptyset}^{\infty} c_j \rho^j$$

$\ell \to \emptyset$

$\rho = \kappa r = \frac{1}{a_0 n r}$, $n \to 1$: $= \frac{1}{a_0 r}$

Σ for $n=1,\ \ell=0,$ $j_{max} = \emptyset$

$$R(r) = \frac{c_0}{a_0} e^{-r/a_0}$$

Normalization: $\int_0^\infty |R(r)|^2 r^2 dr \equiv 1$

$$\Rightarrow c_0 = \frac{2}{\sqrt{a_0}}$$

Also $Y_0^0(\theta, \varphi) = \frac{1}{\sqrt{4\pi}}$

$$\therefore \boxed{\psi_{100}(r, \theta, \varphi) = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}}$$

Degeneracy: How many states have the same energy?

$n =$ positive integer $= 1, 2, 3$

$\ell = 0, 1, 2, \ldots, n-1$

$m = -\ell, -\ell+1, \ldots, \ell-1, \ell.$

Energy depends only on n, so all these states

n	ℓ	m	degeneracy
1	$\varnothing$	$\varnothing$	1
2	$\varnothing$	$\varnothing$	4
	1	$-1, \varnothing, 1$	
3	$\varnothing$	$\varnothing$	9
	1	$-1, \varnothing, 1$	
	2	$-2, -1, \varnothing, 1, 2$	
4	$\varnothing$	$\varnothing$	16
	1	$-1, \varnothing, 1$	
	2	$-2, -1, \varnothing, 1, 2$	
	3	$-3, -2, -1, 0, 1, 2, 3$	

degeneracy $= n^2$

Dirac notation: Let $|n\ell m\rangle$ stand for the hydrogen eigenstates. Then

$$\hat{H}|n\ell m\rangle = \frac{(-13.6\text{ eV})}{n^2}|n\ell m\rangle$$

$$\hat{L}^2|n\ell m\rangle = \hbar^2 \ell(\ell+1)|n\ell m\rangle$$

$$\hat{L}_z|n\ell m\rangle = m\hbar|n\ell m\rangle$$

Orthonormality: $\langle n'\ell' m'|n\ell m\rangle = \delta_{nn'}\delta_{\ell\ell'}\delta_{mm'}$

Completeness: $\sum_{n=1}^{\infty}\sum_{l=0}^{n-1}\sum_{m=-l}^{l} |nlm\rangle\langle nlm| = 1$.

The radial wavefunction are known as the ~~Laguerre~~ "Laguerre Polynomials". They can be found according to

$$\sum_{j=0}^{\infty} c_j \rho^j \equiv L_{n-l-1}^{2l+1}(2\rho)$$

"Associated Laguerre Polynomial"

where $L_{q-p}^{p}(x) \equiv (-1)^p \left(\frac{d}{dx}\right)^p L_q(x)$

"Laguerre Polynomial"

and $L_q(x) \equiv e^x \left(\frac{d}{dx}\right)^q \left(e^{-x} x^q\right)$