Orbital Angular Momentum

In classical mechanics, $\vec{L} = \vec{r} \times \vec{p}$.

In QM, we take $\vec{L} = -i\hbar\, \vec{r} \times \vec{\nabla}$, using $\vec{p} = -i\hbar\vec{\nabla}$

This implies the following commutators:

$$[\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z, \quad [\hat{L}_y, \hat{L}_z] = i\hbar \hat{L}_x, \quad [\hat{L}_z, \hat{L}_x] = i\hbar \hat{L}_y$$

These commutators say that we cannot determine more than one component of $\vec{L}$ at any time.

For example, the U.C. says

$$\Delta L_x \Delta L_y \geq \frac{\hbar}{2}\langle L_z \rangle.$$

$$\Delta L_y \Delta L_z \geq \frac{\hbar}{2}\langle L_x \rangle$$

$$\Delta L_z \Delta L_x \geq \frac{\hbar}{2}\langle L_y \rangle$$

The only way we can know all three components is when $\vec{L} = \emptyset \Rightarrow$ Then $L_x = L_y = L_z = \emptyset$.

On the other hand, we <u>can</u> determine L^2 <u>and</u> <u>any one component of $\vec{L}$</u>, because

$$[\hat{L}_z, \hat{L}^2] = \emptyset, \quad [\hat{L}_x, \hat{L}^2] = \emptyset, \quad [\hat{L}_y, \hat{L}^2] = \emptyset.$$

when $\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$

~~Because~~

Because these commutators are zero, $\hat{L}^2$ & $\hat{L}_z$ have ~~eigenfunctions~~ <u>eigenstates in common.</u>

(Just like $[\hat{H}_{free}, \hat{p}] = \emptyset$ implies that $\hat{H}_{free}$ and $\hat{p}$ have common eigenstates.)

Recall that with the SHO, the commutator relation for $[\hat{a}, \hat{a}^+]$ allowed us to determine the eigenvalues for $\hat{H}_{SHO}$ without solving the Schroedinger Eq. (we found that

$$E_n = \hbar\omega_0 (n + \tfrac{1}{2}), \; n = 0, 1, 2, \ldots$$
$$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle, \quad \hat{a}^+|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

all came from $[\hat{a}, \hat{a}^+] = 1$ & $\hat{H}_{SHO} = \hbar\omega_0 (\hat{a}^+\hat{a} + \tfrac{1}{2})$.

We can use a similar strategy to find the eigenvalues for L^2 & L_z.

First Define $\hat{L}_+ \equiv \hat{L}_x + i\hat{L}_y$, $\hat{L}_- \equiv \hat{L}_x - i\hat{L}_y$ } these will turn out to be similar to $\hat{a}$ & $\hat{a}^+$.

We can calculate from the definitions:

$$[\hat{L}_z, \hat{L}_+] = \hbar \hat{L}_+, \quad [\hat{L}_z, \hat{L}_-] = -\hbar \hat{L}_-$$
&
$$[\hat{L}^2, \hat{L}_\pm] = \emptyset, \quad [\hat{L}_+, \hat{L}_-] = 2\hbar \hat{L}_z$$

One more useful expression: $\hat{L}^2 = \hat{L}_\pm \hat{L}_\mp + \hat{L}_z^2 \pm \hbar \hat{L}_z$

We want to find the eigenvalues for $\hat{L}^2$ & $\hat{L}_z$, using the base states that they have in common.

Let $\{|\ell m\rangle\}$ be the name for these common base states.

(ℓ: a label for the L^2 eigenvalue. m: a label for the L_z eigenvalues)

Then $\hat{L}_z |lm\rangle$ = (eigenvalue for L_z) $|lm\rangle$

Let the eigenvalue be called $\hbar m$, because $\hbar$ has units of angular momentum. We know nothing about m except that it is unitless.

$$\hat{L}_z |lm\rangle \equiv \hbar m |lm\rangle \leftarrow \text{definition of } m.$$

We can show that m is either { an integer, or an odd multiple of 1/2

Like this:

$$\hat{L}_z \left(\hat{L}_+ |lm\rangle \right) = \left(\hbar \hat{L}_+ + \hat{L}_+ \hat{L}_z \right) |lm\rangle$$

using $[L_z, L_+] = \hbar L_+$

$$= \left(\hbar \hat{L}_+ + L_+ (\hbar m) \right) |lm\rangle$$

$$= \hbar (m+1) \left(\hat{L}_+ |lm\rangle \right)$$

$\therefore$ $\hat{L}_+ |lm\rangle$ is an unnormalized eigenstate of $\hat{L}_z$, with eigenvalue $\hbar(m+1)$.

$\therefore$

(It turns out that the normalization constant is

$$\hat{L}_\pm |lm\rangle = \hbar \sqrt{l(l+1) - m(m\pm 1)} \; |l, m\pm 1\rangle$$

Similarly, $\hat{L}_z \left(\hat{L}_- |lm\rangle \right) = \hbar (m-1) \left(\hat{L}_- |lm\rangle \right)$,

so $\hat{L}_- |lm\rangle$ is an unnormalized eigenstate of $\hat{L}_z$ with eigenvalue $\hbar(m-1)$.

So the L_z eigenvalues are separated by one unit of $\hbar$: $\{ \ldots |l, m-2\rangle, |l, m-1\rangle, |l, m\rangle, |l, m+1\rangle \}$

eigenvalues $\hbar(m-2)$, $\hbar(m-1)$, $\hbar m$, $\hbar(m+1)$, ...

Using the $[L^2, L^+]$ commutator, we can also show that

- the eigenvalues for L^2 are $\hbar^2 \ell(\ell+1)$.

  $\Rightarrow \quad L^2 |\ell m\rangle = \hbar^2 \ell(\ell+1) |\ell m\rangle \leftarrow$ e.v. for L^2

- The allowed values of m run from $-\ell$ to ℓ:

  $m = \{-\ell, -\ell+1, \ldots, \ell-1, \ell\}$.

- ℓ is either $\begin{cases} \text{an integer } (\ell = 0, 1, 2, 3, \ldots \\ \text{or half integer } (\ell = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \end{cases}$

  $\Rightarrow$ For orbital angular momentum, (Phys 401) ℓ is an integer

  $\Rightarrow$ For spin angular momentum (Phys 402), ℓ is half-integer

So our orbital angular momentum states are

$\lvert$states$\rangle$	ℓ	L^2	m	L_z
$\lvert 00\rangle$	0	$\varnothing$	$\varnothing$	
$\lvert 1,-1\rangle, \lvert 1,0\rangle, \lvert 1,1\rangle$	1	$2\hbar^2$	$-1, 0, 1$	$-\hbar, \varnothing, \hbar$
$\lvert 2,-2\rangle, \lvert 2,-1\rangle, \lvert 2,0\rangle, \lvert 2,1\rangle, \lvert 2,2\rangle$	2	$6\hbar^2$	$-2, -1, 0, 1, 2$	$-2\hbar, -\hbar, \varnothing, \hbar, 2\hbar$

These states have a property we call degeneracy. When multiple states have the same eigenvalue, we say they are degenerate. For example, $|1,-1\rangle, |1,0\rangle, |1,1\rangle$ all have $L^2 = 2\hbar^2$. We say they are degenerate in L^2. They are not degenerate with respect to L_z, however.

L_z removes the degeneracy

~~Sometimes~~

Orbital Angular momentum

States are $\{|lm\rangle\}$, $l = 0, 1, 2, 3 \ldots$

$m = \{-l, -l+1, \ldots \emptyset \ldots l-1, l\}$

Eigenvalue Equations

$$\hat{L}^2 |lm\rangle = \hbar^2 l(l+1) |lm\rangle$$

$$\hat{L}_z |lm\rangle = \hbar m |lm\rangle$$

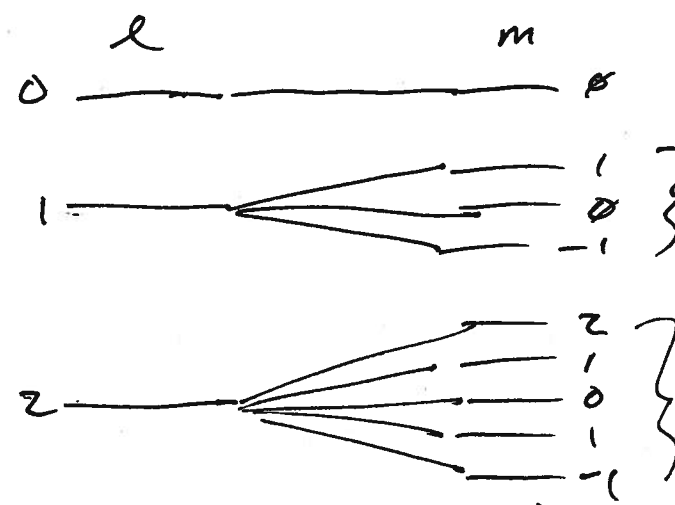
Ladder operators:

$$\hat{L}_\pm = \hat{L}_x \pm i \hat{L}_y$$

$$\hat{L}_\pm |lm\rangle = \hbar\sqrt{l(l+1) - m(m\pm1)}\ |l, m\pm1\rangle$$

↑ promote or demote the L_z eigenv

These states are "degenerate" with respect to $\hat{L}^2$ ⇒ multiple states have the same value of L^2:



These three states all have the same L^2. "triple degeneracy"

"quintuple degeneracy"

Once L_z is specified (m), then the degeneracy is removed".

However, even when L_z is specified, L_x & L_y are still not known with absolute certainty, because L_x, L_y, & L_z do not commute. ⇒ Therefore if we are in an eigenstate of L_z, then L_x & L_y are uncertain:

$$[\hat{L}_z, \hat{L}_x] = i\hbar \hat{L}_y$$, for example, so

$$\Delta L_z \Delta L_x = \frac{\hbar}{2} \langle L_y \rangle$$

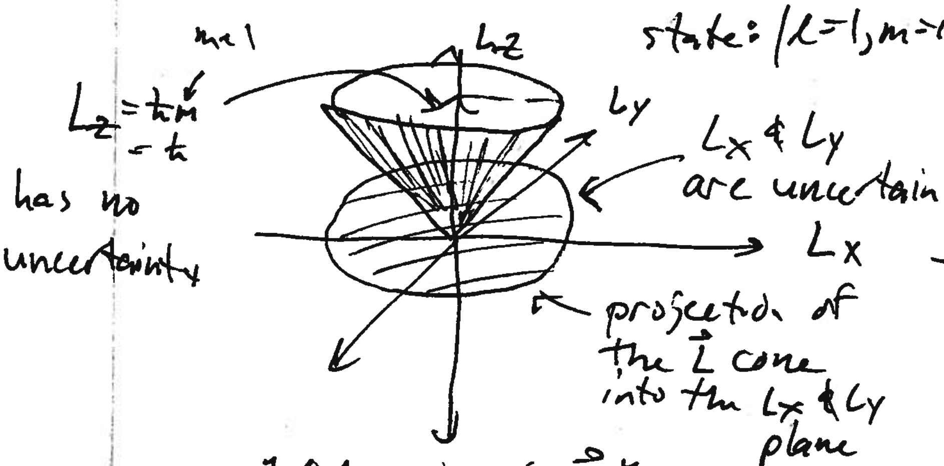
$$[\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z$$, for example, so

$$\Delta L_x \Delta L_y \geq \frac{\hbar}{2} |\langle L_z \rangle|$$ Uncertainty Relation for L_x & L_y.

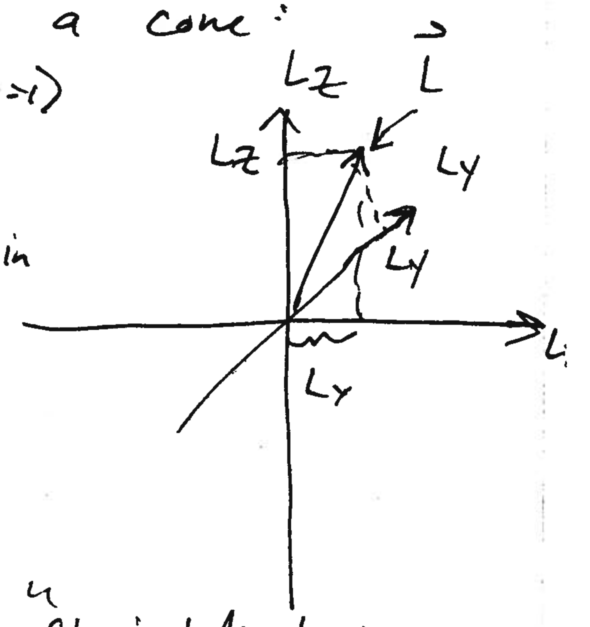
Example: If $|\psi\rangle = |l=1, m=-1\rangle$, then $\langle L_z \rangle = -\hbar$.

Then $\Delta L_x \Delta L_y \geq \left| \frac{-\hbar^2}{2} \right| \geq \frac{\hbar^2}{2}$

We can "picture" these uncertainty relations by picturing the $\vec{L}$ vector as a cone:



"QM view of $\vec{L}$"

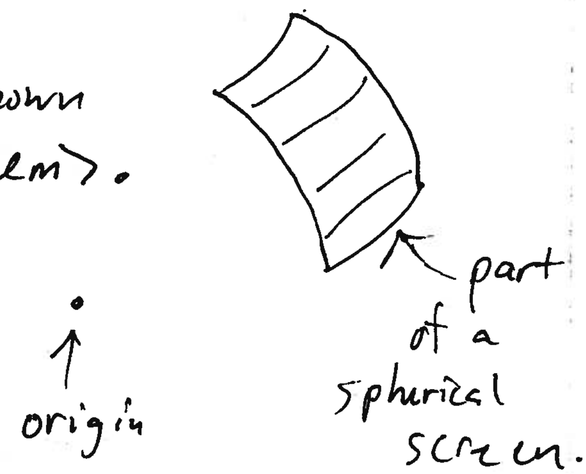


"Classical Mechanics view of L"

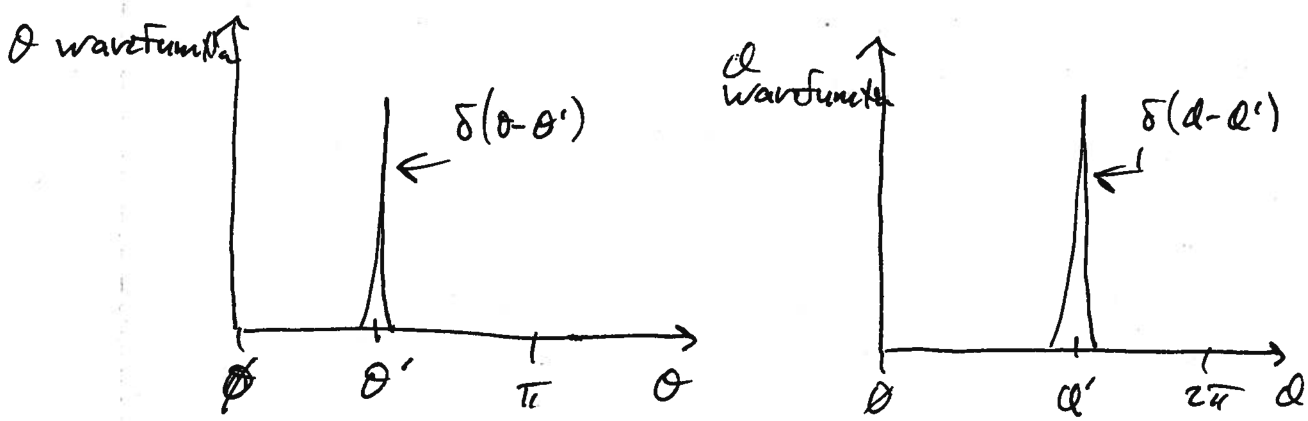
Spatial Wavefunctions for the $\{|lm\rangle\}$

Suppose I have a position measuring device, which measures the θ & ϕ coordinates for a particle. For example, maybe it is a spherical scintillating screen:

Suppose that a particle is known to be in a particular state $|lm\rangle$. Where am I likely to find it on this screen? Where am I unlikely to find it?



We want the amplitude to ~~find~~ observe a particular θ & ϕ for the state $|lm\rangle$. So let $\underline{|\theta\phi\rangle}$ represent a state 100% localized at θ & ϕ:



Then the amplitude to observe $|lm\rangle$ at $|\theta\phi\rangle$ is written $\langle\theta\phi|lm\rangle$ ← QM amplitude to observe θ, ϕ in state $|lm\rangle$

Since θ & ϕ are continuous observables, this is a continuum of amplitudes.

$$\langle\theta,\phi|lm\rangle = \text{continuum of amplitudes} = Y_l^m(\theta,\phi)$$

some complex function

This is analogous to $\langle x|n\rangle = \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}$ for the 1D particle-in-a-box.

So the $Y_\ell^m(\theta,\phi)$ are the spatial wavefunctions for the $\{|lm\rangle\}$. They tell us where we are likely to find the particle, and where we are unlikely.

The Y_ℓ^m are not simple functions. Here is the complete expression:

"Spherical Harmonics" →

$$Y_\ell^m(\theta,\phi) = e^{im\phi}(-1)^m\sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-|m|)!}{(\ell+|m|)!}}\,\underbrace{P_\ell^m(\cos\theta)}_{\text{"Associated Legendre Functions"}},$$

where $P_\ell^m(x) = (1-x^2)^{|m|/2}\left(\frac{d}{dx}\right)^{|m|}\underbrace{P_\ell(x)}_{\text{"Legendre Functions"}}$,

and $P_\ell(x) = \frac{1}{2^\ell \ell!}\left(\frac{d}{dx}\right)^\ell (x^2-1)^\ell$

Explicitly,

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2-1)$

$\vdots$

$Y_0^0(\theta,\phi) = \left(\frac{1}{4\pi}\right)^{\frac{1}{2}}$

$Y_1^1 = -\frac{1}{2}\left(\frac{3}{2\pi}\right)^{\frac{1}{2}}\sin\theta\, e^{i\phi}$

$Y_1^0(\theta,\phi) = \frac{1}{2}\left(\frac{3}{\pi}\right)^{\frac{1}{2}}\cos\theta$

$Y_1^{-1}(\theta,\phi) = \frac{1}{2}\left(\frac{3}{2\pi}\right)^{\frac{1}{2}}\sin\theta\, e^{-i\phi}$

$Y_2^2(\theta,\phi) = \frac{1}{4}\left(\frac{15}{2\pi}\right)^{\frac{1}{2}}\sin^2\theta\, e^{2i\phi}$

$Y_2^1(\theta,\phi) = -\frac{1}{2}\left(\frac{15}{2\pi}\right)^{\frac{1}{2}}\sin\theta\cos\theta\, e^{i\phi}$

$\vdots$