Physics 401 - Homework #3

1) **A square wavefuction (three points each).** Suppose we have a free particle whose wavefuction at t = 0 is described by a square function:

$$\Psi(x,t=0) = \begin{cases} \sqrt{\frac{1}{L}}, \frac{-L}{2} < x < \frac{L}{2} \\ 0, otherwise \end{cases}$$

a) Sketch this wavefunction.

b) Show that this wavefunction is properly normalized.

c) The position operator in quantum mechanics is $\hat{x} = x$. (In other words, the position operator is an instruction to multiply by (x).) Calculate the expectation value for position for this wavefunction.

d) The momentum operator is $\hat{p} = -i\hbar \frac{d}{dx}$. Calculate the expectation value for momentum for this wavefunction.

e) Suppose we change the wavefunction to this:

$$\Psi(x,t=0) = \begin{cases} \sqrt{\frac{1}{L}} \exp(ik_0 x), \frac{-L}{2} < x < \frac{L}{2} \\ 0, otherwise \end{cases}$$

where k_0 is a constant. What is the expectation value of position now?

f) What is the expectation value of momentum now?

2) **A Gaussian wavefunction (three points each).** The Hamiltonian operator for a particle of mass (m) on a spring (a simple harmonic oscillator) is

$$\hat{H} = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}Kx^2,$$

where K is the spring constant.

a) Is the following function an eigenfunction of this Hamiltonian?

$$\varphi(x) = \left(\frac{\alpha}{\pi}\right)^{1/4} \exp\left(-\alpha x^2/2\right), \text{ where } \alpha \equiv \frac{\sqrt{Km}}{\hbar}$$

(Please justify your answer..... a simple yes or no will not be accepted.)

b) Is it an eigenfunction of momentum? (Justify your answer.)

c) Is it an eigenfunction of the position? (Justify your answer.)

3) **Expectation values in stationary states (three points each).** We call the eigenfunctions of the Hamiltonian "stationary states", because in these states the expectation values of all observables are constant in time. We can prove this property as follows.

a) Suppose that the state of a quantum system happens to be an energy eigenfunction at t = 0:

$$\Psi(x,t=0)=\psi(x)=\varphi_n(x)\text{, where } \hat{H}\varphi_n(x)=E_n\varphi_n(x)\,.$$

Write down the complete, time-dependent state function $\Psi(x,t)$ for this system.

b) Write down the quantum mechanical expression for the expectation value of an observable A, using the operator $\hat{A}$, and the state $\Psi(x,t)$.

c) Assume that the operator $\hat{A}$ contains only powers of $\hat{x}$ and $\hat{p}$. Show that under these conditions, the expression from part (b) will be independent of time.

4) **Time evolution of a probability function (three points each).** In problem 3 we showed that all expectation values are constant in time if the system is in a stationary state (an energy eigenfunction). Here we address a related question: does the position probability function, P(x), change in time, or does it also remain constant? Mathematically it is possible that it changes. For example, if the position probability spreads out evenly in the positive and negative directions as time goes forward, then the average value will remain constant.

a) Suppose again that at time t = 0, the state function is an eigenfunction of energy:

$$\Psi(x,t=0)=\psi(x)=\varphi_n(x)\text{, where } \hat{H}\varphi_n(x)=E_n\varphi_n(x)\,.$$

Write down again the fully time-dependent wavefunction $\Psi(x,t)$ for this system.

b) Calculate the position probability distribution P(x,t) for this system. Does it depend on time? Hint #1: the answer is no..... but you should justify this answer! Hint #2: Remember, to calculate P(x,t), you need to multiple Ψ(x,t) by Ψ*(x,t). Make sure you don't multiply Ψ(x,t) by Ψ(x,t)!

c) Now let's suppose that the state function at t = 0 is a sum of two different energy eigenfunctions:

$$\Psi(x,t=0)=\psi(x)=\frac{1}{\sqrt{2}}\varphi_n(x)+\frac{1}{\sqrt{2}}\varphi_m(x)$$

with eigenvalues E_n and E_m. What is the fully time-dependent state function now?

d) Show that the position probability function for this state DOES depend on time, and that the time dependence is described by:

$$P(x,t)=\frac{1}{2}\varphi_n{}^2(x)+\frac{1}{2}\varphi_m{}^2(x)+\varphi_n(x)\varphi_m(x)\cos\left[(\omega_n-\omega_m)t\right],$$

where $\omega_n = E_n/\hbar$ and $\omega_m = E_m/\hbar$. You may assume that φ_n and φ_m are both real functions for this calculation.

The bottom line: The probability distributions for stationary states are constant in time. But if the system is in a superposition of more than one stationary state, then the probability distributions DO evolve in time, due to interference between the various stationary states.