Physics 401 - Homework #2

1) **Two forms of Fourier Series (three points).** If f(x) is periodic, with period 2L, and square integrable between (-L,L), then we can represent it as a linear combination of the functions $\{\exp(in\pi x/L\}$ (a Fourier Series):

$$f(x)=\sum_{n=-\infty}^{n=\infty} c_n e^{in\pi x/L}$$

for some set of coefficients $\{c_n\}$. If f(x) is real, then we can also represent the function this way:

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right)+b_n\sin\left(\frac{n\pi x}{L}\right)\right]$$

where the relationship between the two forms is:

$$a_n = c_n + c_{(-n)}$$
$$b_n = i(c_n - c_{(-n)})$$
$$a_0 = 2c_0$$

Show that these two forms are equivalent by substituting for a_n, b_n, and a_0 in the second form and recovering the first form. Hint: use Euler's formula to convert sines and cosines into exponentials.

2) **Fourier Series of a square wave.** We can write a square wave so it's an odd function:

$$f(x)=\begin{cases}-1, -L<x<0\\ 1, 0<x<L\end{cases}$$, periodic with period 2L.

a) (one point) Sketch this function.
b) (three points) Calculate the Fourier coefficients $\{c_n\}$ for this square wave in the usual way:

$$c_n=\frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}dx.$$

c) (three points) Since this f(x) is real, we can use the sine & cosine form of the Fourier Series. Calculate a_0, $\{a_n\}$, and $\{b_n\}$.
d) (three points) Use a plotting program to graph the Fourier Series on the interval (-3L, 3L) keeping terms up to n = 3.

3) **Ortho-normality condition for Fourier Series**. Show that

$$\frac{1}{2L}\int_{-L}^{L}\left(e^{in\pi x/L}\right)\left(e^{-im\pi x/L}\right)dx=\delta_{nm}\equiv\begin{cases}1, n=m\\ 0, n\neq m\end{cases}$$

by explicitly evaluating the integral for

a) the n = m case (three points).
b) the n $\neq$ m case (three points).

4) **Ortho-normality of Sine functions.** Sine functions also obey an orthonormality condition on the interval (0,L). Show that

$$\frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx=\delta_{nm}$$

by explicitly evaluating the integral for

a) n = m case (three points)
b) the n $\neq$ m case (three points).

Hint: You may use this trigonometric identity: $\sin(u)\sin(v) = \frac{1}{2}\left[\cos(u-v) - \cos(u+v)\right]$.

5) **Fourier Sine Series (three points).** Suppose we have a real function f(x) which we want to represent as a Fourier Series between x = 0 and x = L. If the function goes to zero at x = 0 and x = L, then we will only need the sine terms in the Fourier Series:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right).$$

This is called a Fourier Sine Series. As usual, the question is: what is the correct set of coefficients $\{b_n\}$ for my f(x)? Show that a particular coefficient b_m can be calculated with this rule:

$$b_m = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx.$$

Hint: substitute the Fourier Series representation of f(x) into this expression, then use the ortho-normality condition which you proved in problem #4. (It's the Kronecker Delta (δ_{nm}) which makes this calculation rather simple.)

6) **Fourier Transform of a square pulse.** Yet another square function! This time, the function is a simple square, with no periodicity:

$$f(x) = \begin{cases} 1, \frac{-L}{2} < x < \frac{L}{2} \\ 0, otherwise \end{cases}$$

a) (one point) Sketch this function.

b) (three points) We'd like to represent this function over the entire x-axis (from $-\infty$ to $+\infty$), but the function isn't periodic. This means we can't use the Fourier Series. (By the way, in problem #5, we were able to use Fourier Series because we chose to ignore the series outside the interval (0,L). We're not going to do that here.)

For non-periodic functions represented over the entire x-axis, we must use the Fourier Transform:

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(k)e^{ikx}dk .$$

Now, instead of calculating the discrete coefficients $\{c_n\}$, our job is to calculate the continuous function A(k). Do that for our f(x) using Plancherel's theorem:

$$A(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx$$