Physics 401 - Homework #10

1) We define the quantum mechanical probability current vector $\vec{J}$ in three dimensions in such a way that it satisfies a "continuity equation":

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial P(\vec{r},t)}{\partial t} = 0$$

where $\vec{P}(\vec{r},t)$ is the probability density in three dimensions. (i.e., $P(\vec{r},t)dV$ is the probability to find the particle in a small volume dV.)

a) (three points) The continuity equation implies that the time rate change of the probability to find the particle within any volume V is equal to the negative flux of $\vec{J}$ on the surface of V. Show that this is true, using the divergence theorem from vector calculus. (Roughly speaking, this means that $\vec{J}$ tells us where particle probability is moving across any surface.)

b) (three points) Show that

$$\frac{\partial P(\vec{r},t)}{\partial t} = \frac{i\hbar}{2m}\left[\Psi^* \nabla^2 \Psi - \Psi \nabla^2 \Psi^*\right]$$

Hint: Evaluate the left-hand side using the Born interpretation of the wavefunction, then use the time dependent Schroedinger equation in three dimensions:

$$i\hbar \frac{\partial \Psi}{\partial t} = \left(\frac{-\hbar^2}{2m}\nabla^2 + V(x)\right)\Psi$$

c) (three points) Show that if we define

$$\vec{J} \equiv \frac{\hbar}{2mi}\left(\Psi^* \vec{\nabla} \Psi - \Psi \vec{\nabla} \Psi^*\right)$$

then the equation from part (b) reduces to the continuity equation.

2) The parity operator $\hat{\mathrm{P}}$ is defined in position space such that it reverses the x-coordinate of any function of x:

$$\hat{\mathrm{P}} f(x) = f(-x)$$

The eigenvalue equation for $\hat{\mathrm{P}}$ is

$$\hat{\mathrm{P}} \varphi(x) = \alpha \varphi(x)$$

a) (three points) Show that the eigenvalues of $\hat{\mathrm{P}}$ are (+1) and (-1), and that any purely even or purely odd function is an eigenfunction. Hint: apply the parity operator to both sides of the eigenvalue equation.

b) (three points) Show that the parity operator commutes with any potential function V(x) which is even about x=0. Hint: apply the commutator to an arbitrary function f(x) and show that the result is zero.

c) (three points) Show that the parity operator commutes with the entire Hamiltonian for a particle moving in a potential which is even about x = 0: $[\hat{H},\hat{\mathrm{P}}]=0$.

When the Hamiltonian and the parity operator commute, there are two important consequences:

- **Parity is conserved for these systems.** In other words, the expectation value for parity for these systems will not change as time goes forward, even if the initial state wavefunction is not an eigenstate of parity.
- **The stationary states will be even and odd.** Therefore, for any potential which is symmetric about x = 0, we can classify all the stationary states according to whether they are even or odd. Proof: $\hat{\mathrm{P}}$ and $\hat{H}$ must have some eigenfunctions in common if these operators commute. If there exists a stationary state which is not purely even or odd - call it y(x) - we can create even and odd functions from it by forming the linear combinations y(x) + y(-x) and y(x) - y(-x). These combinations will also be eigenfunctions of the Hamiltonian, so we can always choose the stationary states to be purely even or odd.

d) (3 points) Find the parity of all the particle-in-the-box stationary states. Hint: on Homework #4 we wrote the particle-in-the-box problem in terms of a box centered at x = 0. That form of the states makes the parity easy to find.

3) In class we studied the scattering states of the simple step potential:

$$V(x)=\begin{cases} 0, x\le 0 \\ |V_0|, x>0 \end{cases}$$

where an incoming beam of particles with energy E > $|V_0|$ approaches the potential step travelling from negative (x) towards positive (x). In this problem we will repeat the calculation for the case of E < |V0|.

a) (three points) Write down the Schroedinger equation and its general solution for x < 0 and x > 0.

b) (three points) Identify the term(s) in the general solution which must be zero due to the initial conditions of the scattering experiment.

c) (three points) Write down the boundary conditions that the solution must satisfy.

d) (three points) Determine the reflected amplitude and transmitted amplitude in the general solution in terms of the incoming beam amplitude.

e) (three points) Calculate the magnitude of the transmitted probability current using the 1-D form of the probability current vector.

4) (three points) In Homework #9 we calculated the commutator of the (x) and (y) components of the angular momentum and found: $[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z$. Write down an uncertainty relation for L_x and L_y.

5) (three points) According to problem #4, we cannot in general determine both L_x and L_y to arbitrary accuracy. However, if we know one component of L, say L_z, we can still determine the total angular momentum L^2 with perfect accuracy. Show that this is true by calculating the commutator of L_z and L^2. Hint: use $L^2 = L_x^2 + L_y^2 + L_z^2$, and then use the following commutators:

$$[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z$$
$$[\hat{L}_y, \hat{L}_z] = i\hbar\hat{L}_x$$
$$[\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y$$

Once again, there is no need to use the explicit form of these operators for this problem. All of the relevant information is contained in the above commutator relations.