For Light, we have two physical theories

<u>Geometric Optics</u>: light travels along rays which reflect & refract

<u>Physical Optics</u>: light is a wave in the EM field.

Geometric Optics apply when wavelengths are very short,
Physical Optics is more general and contains geometric optics

For matter we have

<u>Classical mechanics</u>: matter composed of particles which accelerate in response to forces.
but we also have

<u>Quantum mechanics</u>: matter is fundamentally a wave phenomenum

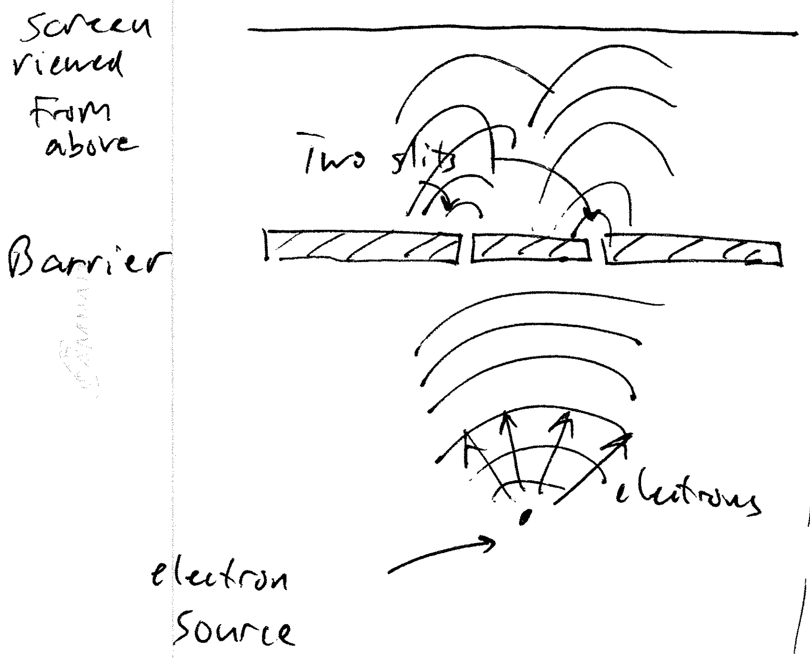
Quantum Mechanics generalizes CM in a manner similar to the way that PO generalizes GO.

However, some aspects ~~ar~~ of QM are fundamentally different than classical waves ~~like~~ theories.

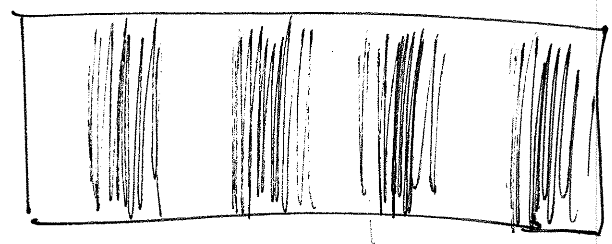
- the role of probability
- nature of identical particles
- entangled states
- the nature of measurement.
- "internal degrees of freedom" → spin.

In Physics 401 we study probability & measurement.
Entangled States & Identical particles is mostly Phys. 402

Wave nature of matter, Two Slit experiment

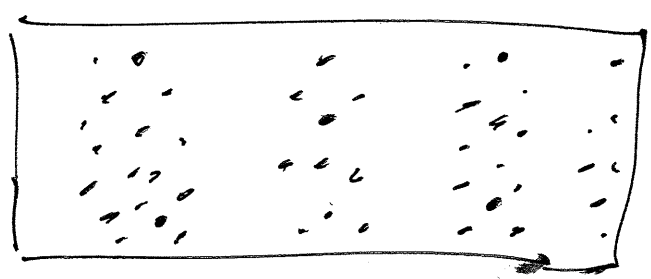


What we see on the screen:
Interference pattern.



electrons traveling into the board

Now Turn down the intensity of the electron source.
Now we see on the screen:



- Electrons arrive one at a time, hitting only one spot ⟹ The screen detects each electron as a single particle.
- The arrival ~~dist~~ locations, in total, exhibit an interference pattern.
- The interference shows up in the probability distribution
- Each electron "interferes" with itself, not other electrons, because only one electron is in the apparatus at the same time.

~~In everyday language~~

But which slit does ~~the~~ each electron travel through?

~~If we try to meas~~

We can measure the location of the electron when it exits the slit by shining a light on the electron at the barrier. If we do this, we know which slit each electron goes through, but ~~then~~ bouncing the light off the electron disturbs it such that the interference pattern on the screen dissappears.

> We ~~can~~ never observe an interference pattern on the screen in the same experiment where we know which slit the electron passes through.

General Principles of Two-Slit experiment, in everyday language

1) The two slit barrier sees the electron as a wave ~~Matter propagates as a wave.~~
2) The screen observes each electron as a <u>single particle</u> → highly localized at one spot.
3) If we measure the electron location at the two slit barrier, we disturb the electron. we no longer see interference on the screen.

General Principles of Two Slit experiment in the language of Quantum Mechanics

1) Electron propagation is described by a "wavefunction" ~~which can be calculated using~~ QM.

2) The wavefunction describes only the probability of finding the electron at each location on the screen, ~~but does not predict the electron trajectory~~ Nothing else about the electron's path can be known.

3) The screen measures the electron location. This location measurement causes the wavefunction to "collapse" to a single point.

~~4) If we measure the electron~~

4) When the electron passes through the barrier, it is in a "superposition of states" → it passes through both slit # 1 & slit # 2.

5) If we measure exactly which ~~st~~ intermediate state it is in → slit 1 or slit 2 → then the measurement causes the wavefunction to collapse, so no interference will be seen on the screen.

In the late 1920's, Einstein & Bohr ~~argued~~ debated about the interpretation of QM. Einstein believed that

- the electron has a true location at all times, whether or not you measure it. ⇒ "Do you really believe that the moon is only there when you look at it?"
- the probabilities in QM reflect the fact that we do not ~~measure~~ know the value of some "hidden variable" which determines the electron's path. ⇒ "God does not play dice"!
- that QM should be replaced by a more complete theory which includes the hidden variable, and that this theory would be deterministic. ⇒ no fundamental probabilities

In 1935, Einstein, Podolsky, & Rosen showed that the collapse of the wavefunction must be instantaneous everywhere in the universe, which appears to violate special relativity.

- Einstein concluded the QM is fundamentally flawed.
- Einstein believed a more complete theory would be "local" (no violation of S.R.) and deterministic.

In 1962, John Bell showed that any local, deterministic theory must make some experimental predictions which are different from QM.

In 1982, Alain Aspect showed that QM is verified experimentally.

→ No local, deterministic theory can describe the universe.

· Most physicists today accept that the universe is fundamentally probabilistic, and that locality is violated.

· If Einstein were alive, he would almost certainly admit that he must give up either determinism, locality, or both.

Complex numbers

Complex numbers can be thought of as a pair of numbers, ~~one real, and one imaginary.~~

$$z = (x, y)$$

Addition is defined by

$$z_1 + z_2 = (x_1 + x_2,\ y_1 + y_2)$$

And multiplication by

$$z_1 z_2 = (x_1, y_1)(x_2, y_2) = (x_1 x_2 - y_1 y_2,\ y_1 x_2 + x_1 y_2)$$

The multiplication rule is easier to remement if you think of $i = \sqrt{-1}$, and $z_1 = x_1 + i y_1$,

$$z_2 = x_2 + i y_2.$$

So we usually don't use the (x, y) notation. It is a very common operation to reverse the sign of the imaginary component → this is complex conjugation.

complex conjugation.

$$z^* = (x + iy)^* = x - iy.$$

For example, $zz^* = (x+iy)(x-iy) =$

$$= x^2 + y^2 + i(yx - xy)$$

$$= x^2 + y^2$$

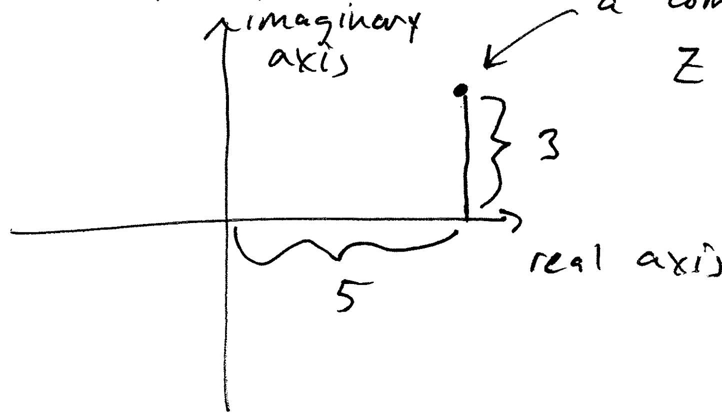
So if $z = 5 + i3$, $z^* = 5 - i3$,

and $zz^* = 25 + 9 = 34$.

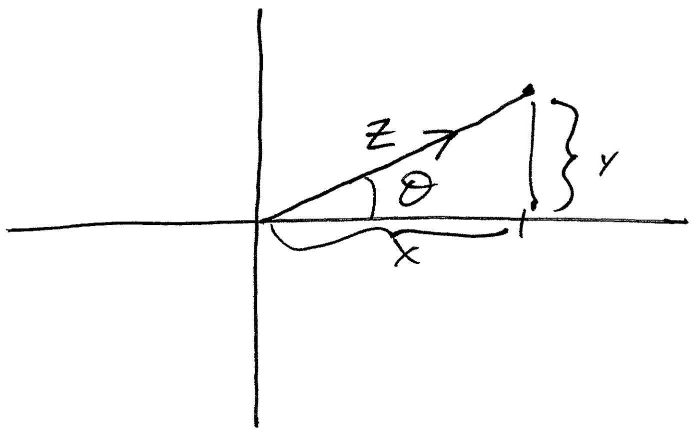
Division: If a complex number is in the denominator, multiple both top & bottom by the conjugate: $\frac{1}{x+iy} = \frac{x-iy}{(x+iy)(x-iy)} = \frac{x-iy}{x^2+y^2}$

We can also think of complex numbers <u>geometrically</u> in the complex plane.

a complex number, $Z = 5 + i3$



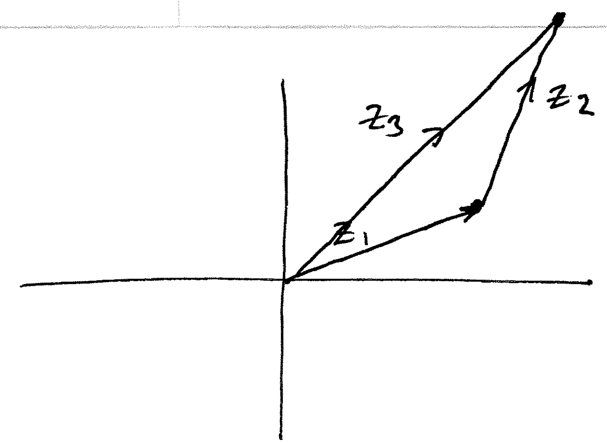
We can also think of a vector from the origin to the complex number



The length of this vector is $|Z| = \sqrt{x^2+y^2} = \sqrt{ZZ^*}$.
The angle between the real axis and the complex number is $\tan\theta = \dfrac{y}{x}$

The angle θ is called the phase of the complex number.

Geometrically, we can add complex numbers in the plane just like vectors:



Can we define the cosine function for a complex number? $\cos(z) = ???$

Yes, using a Taylor Series. For real numbers,

$$\cos(x) = 1 - \tfrac{1}{2}x^2 + \tfrac{1}{4!}x^4 + \cdots$$

So we can define

not zz^*, simply $z^2 = zz$

$$\cos(z) = 1 - \tfrac{1}{2}z^2 + \tfrac{1}{4!}z^4$$

Then, for example,

$$\cos(i) = 1 - \tfrac{1}{2}(i)^2 + \tfrac{1}{4!}(i)^4 + \ldots = 1 + \tfrac{1}{2} + \tfrac{1}{4!} + \cdots$$

How about exponentials? Again, use Taylor Series.

$$e^x = 1 + x + \tfrac{1}{2!}x^2 + \tfrac{1}{3!}x^3 + \cdots$$

So we define

$$e^z = 1 + z + \tfrac{1}{2!}z^2 + \tfrac{1}{3!}z^3 + \cdots$$

Then the exponential of a purely imaginary number $z = i\theta$ (no real part) is

$$e^{z} = e^{i\theta} = 1 + i\theta + \frac{1}{2!}(i\theta)^2 + \frac{1}{3!}(i\theta)^3 + \dots$$

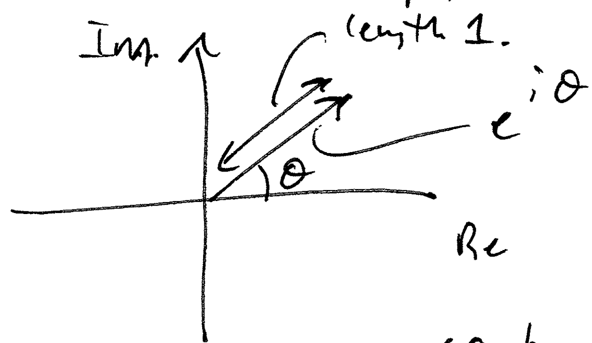
$$= \underbrace{\left(1 - \frac{1}{2}\theta^2 + \frac{1}{4!}\theta^4 + \dots\right)}_{\text{Taylor series for } \cos\theta} + i\underbrace{\left(\theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 + \dots\right)}_{\text{Taylor series for } \sin\theta}$$

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$ Euler formula

So the Real part of $e^{i\theta}$ is $\cos\theta$
& the imaginary part of $e^{i\theta}$ is $\sin\theta$.

The magnitude of $e^{i\theta}$ is $|e^{i\theta}| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$.
And the phase is $\text{tangent}(\text{phase of } e^{i\theta}) = \frac{\sin\theta}{\cos\theta} = \tan\theta$

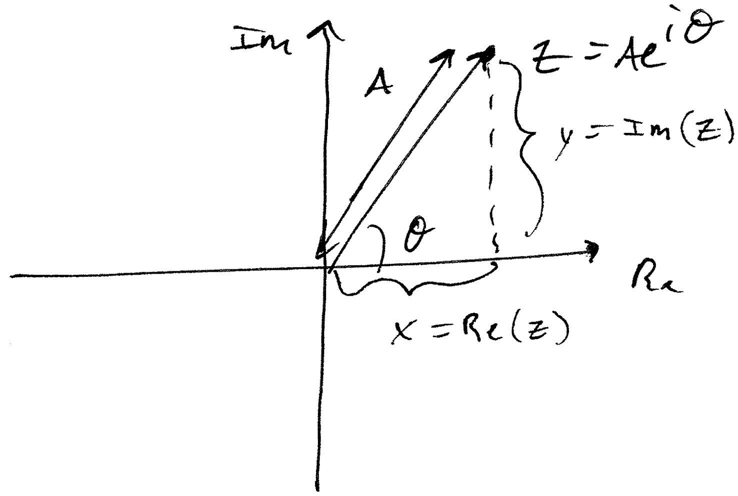
So the θ in $e^{i\theta}$ is simply the phase of $e^{i\theta}$:



Any point in the complex plane can be represented by

$$z = A e^{i\theta}$$

a real number which scales the unit vector $e^{i\theta}$.



A complex number has magnitude A and phase θ. It has real component x and imaginary component y.

$$z = Ae^{i\theta} = x + iy$$

$$A = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}(y/x)$$

$$x = A\cos\theta$$

$$y = A\sin\theta.$$

One nice consequence of Euler's formula is that trigonometric formulas become easy to derive.

Example: prove that $\cos(\theta_1 + \theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$.

Complex numbers are useful for representing waves in classical mechanics. In that case, we always take the real part of the answer at the end of the calculation to get the physical wave.

In QM, however, complex numbers are <u>fundamental</u>.

QM cannot be formulated without them so complex numbers are just as physical as real numbers.

Differential Equations

Equations such as $y^2 - y - 2 = 0$

and $\cancel{\cos(y) = e^{y}}$

$$y^3 + \frac{1}{y} = 57$$

are called algebraic equations. When we find a value of y that satisfies the equation, we say that y is a solution. For the 1st equation, $y = 2$ and $y = -1$ are solutions.

Finding the solution to an algebraic can be very difficult, but checking a proposed solution is very easy (just plug it in!). For an typical algebraic equation, the only method that we have to solve it is to guess a solution, check it, and then modify the guess based on the result.

Differential Equations are similar to algebraic ones, but the solutions are functions. Examples:

$$\frac{dF(t)}{dt} = F^2(t)$$

$$\frac{d^2y(x)}{dx^2} = Ay(x)$$

$$\cancel{\frac{d^2 y(x)}{dx^2} = A\, y(x)}$$

$$\frac{d^2 y(x)}{dx^2} = \sin(\omega x)$$

Finding the solution to a D.E. can be very hard, but it is easy to check a proposed solution $\Rightarrow$ just plug it in.

> The best way to solve a D.E. is to guess a solution and try it out.

On the other hand, it is easy to know how many free parameters the D.E. should have, and how many initial conditions we need to find a specific solution:

> A D.E. ~~with a~~ which has a maximum of (n) order derivatives has (n) free parameters and requires (n) initial conditions. For a unique solution.

Most D.E's. can only be solved numerically, or maybe by a power series method.

But a few D.E.s are so simple & common that everyone should know the solution right away.

1) $\frac{dy(t)}{dt} = A$ $\Rightarrow$ Sol'n: $y(t) = At + C$.

1st order

one free parameter

2) $\frac{dy(t)}{dt} = Ay(t)$ $\Rightarrow$ Sol'n: $y(t) = Ce^{At}$

1st order

one free parameter

3) $\frac{d^2y(t)}{dt^2} + Ay(t) = \emptyset$ $\Rightarrow$ Many ways to write this solution:

$\Rightarrow$ Sol'n: $y(t) = C_1 \sin(\sqrt{A}t + C_2)$

or $y(t) = C_2 \cos(\sqrt{A}t + C_2)$

or $y(t) = C_1 \sin(\sqrt{A}t) + C_2 \cos(\sqrt{A}t)$

or $y(t) = C_1 e^{i\sqrt{A}t} + C_2 e^{-i\sqrt{A}t}$

or $y(t) = C_1 \sinh(\sqrt{-A}t) + C_2 \cosh(\sqrt{-A}t)$

or $y(t) = C_1 \sinh(\sqrt{-A}t + C_2)$

or $y(t) = C_1 \cosh(\sqrt{-A}t + C_2)$

C_1 & C_2 will change from one solution to the next!

When an initial condition(s) is specified, the constant(s) can be fixed.

For example:

$$\cancel{d^2x}\quad \frac{d^2y(t)}{dt^2} + 4y(t) = 0$$

General solution: Choose cosine form for convenience, since A is positive.

$$y(t) = C_1 \cancel{A} \cos\left(\sqrt{4}\,t + C_2\right) = C_1 \cos(2t + C_2)$$

Suppose we are told $y(0) = 1$ and $\frac{dy(0)}{dt} = 0$.

Then

$$y(0) = C_1 \cos(C_2) = 1 \qquad (1)$$

and $$\frac{dy(t)}{dt} = -2C_1 \sin(2t + C_2) = 0 \text{ at } t \to 0$$

$$\text{or } +C_1 \sin(\cancel{2t} + C_2) = 0 \qquad (2)$$

$C_1 = 0$ would satisfy (2) but violate (1). So we must have $C_1 \neq 0$, so that

$$\sin(C_2) = 0 \implies C_2 = 0, \pi, 2\pi, \ldots$$

take this one

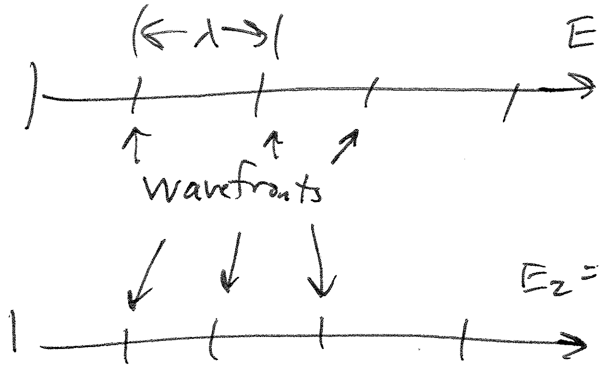
Then (1) tells us that $C_1 = 1$.

So $\boxed{C_1 = 1,\ C_2 = 0}$ Gives the specific solution.

$$\boxed{y(t) = \cos(2t).}$$

Lecture Demo: M1-11 Laser Diffraction & P2-11 Interference of Photons.
Interference of Light – Two Slit experiment

Suppose we have two waves:



$E_1 = A_0 \sin \omega t$

$E_2 = A_0 \sin(\omega t + \delta)$

in general there could be a phase difference between wave 1 & wave 2

waves overlap here

• p

Electric Field at point P:

$$E_p = E_1 + E_2 = A_0\left[\sin(\omega t) + \sin(\omega t + \delta)\right]$$

Trig Identity: $\sin\alpha + \sin\beta = 2\cos\left(\frac{\alpha-\beta}{2}\right)\sin\left(\frac{\alpha+\beta}{2}\right)$

$$E_p = \underbrace{2A_0 \cos\left(\frac{\delta}{2}\right)}_{\text{"amplitude factor"}} \underbrace{\sin\left(\omega t + \delta/2\right)}_{\text{"wave factor"}}$$

- oscillating 10^{14} times per second for optical light!
- Too fast for the human eye to see.
- We see the average.

$$\text{Intensity on screen} \propto E^2 = \left[4A_0^2\cos^2\left(\frac{\delta}{2}\right)\right]\cancel{\sin^2(\omega t + \delta)}$$

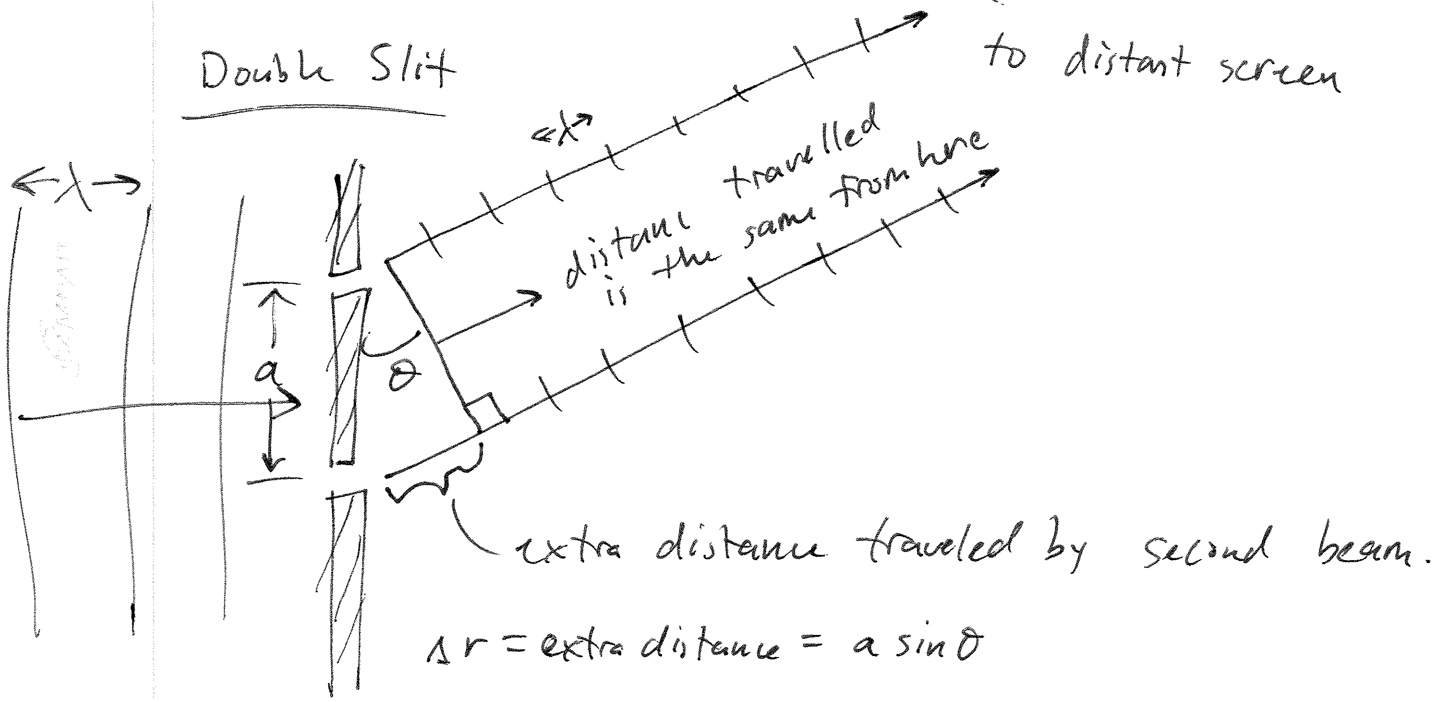
unobserved in visible light

$$\underbrace{\text{Intensity} = 4A_0^2\cos^2\left(\frac{\delta}{2}\right)}_{\text{How the intensity depends on the phase difference.}}$$

Constructive Interference when $\delta = \pm m 2\pi$, $m = 0, \pm 1, \pm 2, \ldots$

Destructive Interference when $\delta = \left(m + \frac{1}{2}\right) 2\pi$ $m = 0, \pm 1, \pm 2, \ldots$

Double Slit



Δr = extra distance = $a \sin\theta$

If $\Delta r = \lambda$, then the phase difference δ is 2π.

so $\dfrac{\Delta r}{\lambda} = \dfrac{\delta}{2\pi}$

In general $\delta = \left(\dfrac{2\pi}{\lambda}\right)\Delta r = k(\Delta r)$

where $k \equiv 2\pi/\lambda$ = wavenumber

So intensity on the screen as a function of θ is

$$I = 4A_0^2 \cos^2\frac{\delta}{2} = 4A_0^2\cos^2\left(\frac{k(\Delta r)}{2}\right)$$

$$= 4A_0^2\cos^2\left(\frac{ka\sin\theta}{2}\right)$$

Wave Formula sheet and Dictionary

λ = wavelength (meters)

K = "wavenumber" $= \dfrac{2\pi}{\lambda}$ $\left(\dfrac{1}{\text{meters}}\right)$

T = period (seconds)

ν = frequency $\left(\dfrac{1}{\text{seconds}} \text{ or Hz}\right) = \dfrac{1}{T}$

ω = angular frequency $= \dfrac{2\pi}{T} = 2\pi\nu$ (Hz)

Phase Difference due to a path length difference

$$= K(\Delta r) \quad \text{or} \quad Kx \quad \text{or} \quad 2\pi\left(\frac{x}{\lambda}\right)$$

Phase due to a "displacement in time"

$$= 2\pi\left(\frac{t}{T}\right) \quad \text{or} \quad \omega t$$

Planck formula: Light wave composed of photon.

Energy of each photon is $E = h\nu = \left(\dfrac{h}{2\pi}\right)(2\pi\nu) = \hbar\omega$

frequency

Planck's Constant $= 6.63 \times 10^{-34}$ J·s

$\hbar \equiv \dfrac{h}{2\pi}$

DeBroglie: ~~Matter~~ All particles both matter and light have a wavelength:

DeBroglie wavelength $= \lambda = \dfrac{h}{p}$ ← particle momentum.

For photons, momentum $= p = \dfrac{E}{c}$ from special relativity.

~~or~~ so $\cancel{\lambda}$ $\dfrac{h}{p} = \dfrac{h}{E/c} = \left(\dfrac{h}{E}\right)c = \dfrac{c}{\nu} = \lambda$

If $\lambda = \frac{h}{p}$ then $p = \frac{h}{\lambda} = \left(\frac{h}{2\pi}\right)\left(\frac{2\pi}{\lambda}\right) = \hbar k$

Summarizing:

$$E = h\nu = \hbar\omega$$

$$p = \hbar k$$

Expectation Values & Variance

Consider an observable quantity C. We measure C many times and get results:

numerical result	# of times we measured it
c_1	n_1
c_2	n_2
c_3	n_3
c_4	n_4
$\vdots$	n_5
$\vdots$	n_i
c_K	n_K

Define N as the total # of measurements: $N = \sum_{i=1}^{K} n_i$

The mean or average value of C is

$$\langle C \rangle \equiv \overline{C} = \frac{1}{N}\sum_{i=1}^{K} n_i c_i = \sum_{i=1}^{K}\left(\frac{n_i}{N}\right)c_i = \sum_{i=1}^{K}(P_i)\,c_i$$

where $P_i \equiv$ Probability of result c_i

$$P_i \equiv \frac{n_i}{N}.$$

In QM we call the mean or average value the "expectation value"

If C is a continuous quantity (like position x), we use an integral: $\langle C \rangle = \int_{\text{all possible } C\text{'s}} C\, P(C)\, dC$

Ex: Expectation of position x: $\langle x \rangle = \int_{\text{all possible } x\text{'s.}} x\, P(x)\, dx$

Standard deviation / variance of C:

If value c_i differs from $\langle C \rangle$ by amount $\delta_i \equiv c_i - \langle C \rangle$, and we square each δ_i, and calculate the mean δ_i^2, we get the variance:

some people use — some people use

$$\text{variance} \equiv \sigma^2 \equiv (\Delta C)^2 \equiv \langle (\delta)^2 \rangle$$
$$= \frac{1}{N} \sum_{i=1}^{K} \left(c_i - \langle c \rangle\right)^2 n_i$$

Griffiths

Theorem: (Proof in Griffiths ?):

$$(\Delta c)^2 = \langle c^2 \rangle - \langle c \rangle^2$$

square each c_i, then average

find the average of c_i, then square.

this result often makes $(\Delta C)^2$ easier to calculate.

The Standard Deviation is the square root of the variance.

$$\text{std. dev.} \equiv \sqrt{(\Delta C)^2} = \Delta C = \sqrt{\sum_{i=1}^{K} \frac{1}{N} \left(c_i - \langle c \rangle\right)^2 n_i}$$

or

$$\Delta C = \sqrt{\langle c^2 \rangle - \langle c \rangle^2}$$

The Uncertainty Principle of QM is formulated in terms of standard deviations.

The std. dev. measured how wide the distribution of $\{c_i\}$ are. Lots of variation in $\{c_i\}$? $\Rightarrow$ large ΔC.
Minimal variation in $\{c_i\}$? $\Rightarrow$ small ΔC.

Postulate I of QM - Eigenvalues and EigenFunctions.

For any observable A, there is an

- operator $\Rightarrow \hat{A}$ ← hat means operator
- eigenvalue(s) $\Rightarrow a$ ← a numerical value
- eigenfunction(s) $\Rightarrow \phi$ ← a function.