Physics 401 - Homework #5

1) **A quantum mechanical state in two bases (12 points total).** A quantum mechanical particle-in-a-box is in a superposition of two stationary states, the n = 4 state and the n = 5 state. The superposition is an equal mixture of the two states, but the n = 5 state is out-of-phase by an angle of π compared to the n = 4 state. We can represent this state by a ket-vector:

$$\left| our-favorite-QM-state \right\rangle,$$

but this symbol doesn't tell us much about the state. We can, however, write the state in other forms which are more explicit, and more useful for doing calculations. For example, we can project the state into the position basis, or we can project it into the energy basis.

a) (2 points) Write down an explicit representation of this state in the position (x) basis. In other words, write down the wave function $\psi(x)$ for this state.

b) (2 points) Write down an explicit representation for this state in the energy basis. In other words, write down the $\{a_n\}$ for this state in a column vector format.

c) (2 points) What is the energy basis good for? To answer this, name one type of question that requires almost no calculation to answer in the energy basis. Also name a second type of question that is similarly easy to answer in the position basis.

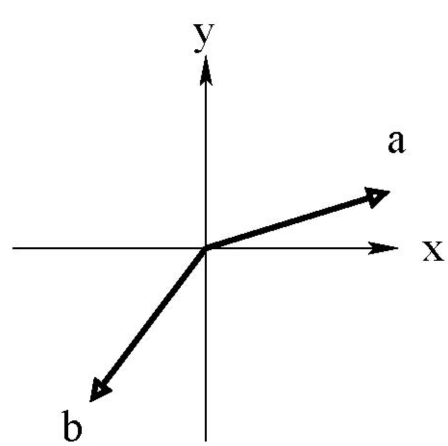
d) (3 points) Calculate the overlap between this state and the state from Homework #3, question 2, working in the energy basis. In other words, calculate

$$\left\langle the-QM-state-for-Homework-3-question-2 \middle| our-favorite-QM-state \right\rangle$$

using the column-vector and row-vector format of the energy basis. You may use the solution for Homework #3, posted on the course website.

e) (3 points) Write down an expression for the overlap of these two states using the position (x) basis. Evaluate this expression.

2) **Dot-product of ordinary vectors (eight points total).** *To answer this question, you will need a hard copy of this homework assignment.* Two ordinary vectors $\vec{a}$ and $\vec{b}$ are shown in the figure below.



a) (2 points) Draw a sketch of this diagram into your homework. Add to your sketch the projections of vectors $\vec{a}$ and $\vec{b}$ onto the (x) and (y) axes. Also draw these projections on the hardcopy of this homework assignment.

b) (2 points) On the hardcopy of this assignment, use a ruler to measure the lengths of the (x) and (y) components for each vector **in units of inches**, and write down the explicit expression for each vector in a row vector format: (a_x,a_y), (b_x, b_y).

c) (2 points) Calculate the squared length of each of these vectors **in units of (squared) inches** using the dot product rule:

$$|\vec{v}|^2 = \vec{v}\cdot\vec{v} = \sum_i v_i v_i = v_x v_x + v_y v_y$$

Confirm that your result is correct by measuring the length of each vector with your ruler.

d) (2 points) Calculate the dot product of these two vectors using this rule:

$$\vec{a}\cdot\vec{b} = \sum_i a_i b_i = a_x b_x + a_y b_y$$

Note: to calculate the dot product, please **do not** use this common formula: $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos(\theta)$. This second form is correct, of course, but it doesn't make a good analogy with the Dirac bracket in quantum mechanics.

Here's the point: By projecting the vectors onto the (x) and (y) axes, you essentially calculated the dot product in this way:

$$\vec{a}\cdot\vec{b} = a_x b_x + a_y b_y = (\vec{a}\cdot\hat{x})(\hat{x}\cdot\vec{b}) + (\vec{a}\cdot\hat{y})(\hat{y}\cdot\vec{b}) = \sum_i (\vec{a}\cdot\hat{i})(\hat{i}\cdot\vec{b}),$$

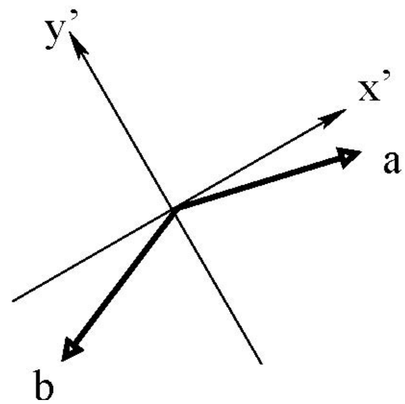
$$\therefore \vec{a}\cdot\vec{b} = \sum_i (\vec{a}\cdot\hat{i})(\hat{i}\cdot\vec{b})$$

where $\hat{i}$ is $\hat{x}$ or $\hat{y}$. *This is exactly analogous to the way that we calculate "brackets" for bra and ket vectors in quantum mechanics:*

$$\langle a|b\rangle = \langle a|\left(\sum_n |n\rangle\langle n|\right)|b\rangle = \sum_n \langle a|n\rangle\langle n|b\rangle$$

In this expression, we have inserted the identity operator $\hat{I} = \sum_n |n\rangle\langle n|$ between $\langle a|$ and $|b\rangle$. By doing this insertion, we are projecting these bra and ket vectors into a particular basis. Make sure that you understand this expression and the analogous expression above for ordinary vectors.

3) **Dot product in a different coordinate system (three points).** In the diagram below, the vectors $\vec{a}$ and $\vec{b}$ are shown again, but this time we've chosen to use a different coordinate system.



Repeat your calculation of the dot product of $\vec{a}$ and $\vec{b}$ by projecting them onto this new coordinate system, measuring with your ruler, and using the sum-over-components formula for the dot product given in question 2(d). **Again, please measure in inches.**

Here's the point: You can choose to work in any coordinate system that you like, but the dot product of $\vec{a}$ and $\vec{b}$ is independent of that choice, so you should get the same answer in questions 2(d) and 3. Quantities that are independent of coordinate system are called scalars, and the dot product is a scalar. The vector components a_x, a_y, b_x, and b_y, are not scalars, because they change depending on which coordinate system you use. In quantum mechanics, we can choose to calculate a Dirac bracket in any representation we choose, but the result should be independent of our choice:

$$\langle a|b\rangle = \sum_n \langle a|n\rangle\langle n|b\rangle = \sum_n a_n^* b_n$$

and

$$\langle a|b\rangle = \int \langle a|x\rangle\langle x|b\rangle dx = \int \psi_a^*(x)\psi_b(x)dx$$

Physically, the result must be independent of our choice of basis, because $\langle a|b\rangle$ is the same quantum mechanical amplitude, no matter how we go about calculating its value.

4) **Identities with bra and ket vectors (two points each).** "Prove" the following identities for bra and ket vectors. For parts (a), (c) and (d), you may work in the column-vector and row-vector format of the energy basis. (Disclaimer: most of these identities we effectively assumed to be true when we first defined our Dirac notation, so your proof will be somewhat circular in a rigorous sense. "Prove" them anyway.)

a) $\langle a|b\rangle^* = \langle b|a\rangle$

b) Use part (a) to show that $\langle a|a\rangle$ is always a real number.

c) $\alpha(|a\rangle + |b\rangle) = \alpha|a\rangle + \alpha|b\rangle$, for any complex number α.

d) $\langle c|(|a\rangle + |b\rangle) = \langle c|a\rangle + \langle c|b\rangle$

e) If $|a\rangle = \alpha|b\rangle$, then $\langle a| = \alpha * \langle b|$. Hint: Multiply the first expression by $\langle a|$, and then use the identities from parts (a) and (b).

5) **Dot product of bra and ket vectors (two point each)**.

a) Evaluate $\langle a|b\rangle$, where $|a\rangle = \alpha|1\rangle + \beta|2\rangle$ and $|b\rangle = \gamma|1\rangle + \delta|3\rangle$, and α, β, γ, and δ are complex numbers.

b) Write down a normalization condition that α and β must satisfy if $|a\rangle$ is a normalized state.