

Physics 401 - Homework #3

1) **Expectation values in stationary states (three points).** When the state of a quantum mechanical system is an eigenfunction of the Hamiltonian, the expectation values of all observables are constant in time. For this reason we refer to these states as "stationary". We can prove this property as follows.

a) (1 point) Suppose that the state of a certain quantum system happens to be an energy eigenfunction at $t = 0$:

$$\Psi(x, t = 0) = \psi(x) = \varphi_n(x), \text{ where } \hat{H}\varphi_n(x) = E_n\varphi_n(x).$$

Write down the complete, time-dependent state function $\Psi(x, t)$ for this system.

b) (1 point) Write down the quantum mechanical expression for the expectation value of an observable A , using the operator \hat{A} , and the state $\Psi(x, t)$.

c) (1 point) Show that as long as the operator \hat{A} contains only powers of \hat{x} and \hat{p} , then the expression from part (b) will be independent of time.

2) **Particle-in-box wavefunction (eight points).** A particle in a one dimensional box of length L is in the following state at $t = 0$:

$$\Psi(x, t = 0) = \psi(x) = A \sin\left(\frac{5\pi x}{L}\right) \cos\left(\frac{\pi x}{L}\right)$$

a) Use a trigonometric identity to re-write this wavefunction as a sum of normalized energy eigenfunctions for this system. Be sure to show the coefficient for each normalized eigenfunction explicitly.

b) Calculate the normalization constant (A), and determine its units. Hint: once the wavefunction is written as a sum of normalized eigenfunctions, it should be easy to evaluate the appropriate integral which determines (A) using the ortho-normality condition.

c) What are the possible results of an energy measurement performed on this system at $t = 0$?

d) What are the probabilities that each of those results occur?

e) What is the expectation value for the energy of this system at $t = 0$? Hint: do not attempt to evaluate an integral for this calculation; use your previous results to get the answer quickly.

f) Write down the time-dependent wavefunction for this system.

g) Calculate $P(x, t)$, the probability distribution of the position of the particle as a function of time.

h) Suppose we measure the energy of this system at $t = 0$. Choose as the result of the measurement one of the values that you wrote down in part (c), and write down the corresponding wavefunction for the system immediately following the measurement.

3) **Free-particle square wavefunction (ten points).** Suppose we have a free particle whose wavefunction at $t = 0$ is described by a top-hat function:

$$\Psi(x, t = 0) = \begin{cases} \sqrt{\frac{1}{L}}, & -\frac{L}{2} < x < \frac{L}{2} \\ 0, & \text{otherwise} \end{cases}$$

We can think of this wavefunction as being created by a plane wave when it meets a barrier which has a single slit of width L . In fact, our analysis below describes the diffraction that happens on the far side of the slit, although we will keep things simpler by treating the problem in one dimension only.

- a) Sketch this wavefunction.
- b) Show that the wavefunction in momentum space is given by

$$\phi(k) = \sqrt{\frac{2}{\pi L}} \frac{\sin(kL/2)}{k}$$

(In other words, take the Fourier transform of $\Psi(x,t=0)$.)

- c) What are the units of Ψ and ϕ ?
- d) Sketch $\phi(k)$ and $|\phi(k)|^2$. You may use a computer if you wish. Note: the expression for $\phi(k)$ appears to have a singularity at $k = 0$, but you may ignore this. (In fact, the function is finite and continuous at $k=0$, as can be seen by taking the limit with l'Hopital's rule.)
- e) Certain values of the momentum are forbidden for this wavefunction. That is, if we perform a momentum measurement, certain values have zero probability of occurring. What are those values?
- f) Looking at $|\phi(k)|^2$, what is the expectation value for the momentum? (You don't need to evaluate any integrals for this question, as long as you explain your reasoning.)
- g) Write down an expression for $\Psi(x,t=0)$ as an integral over $\phi(k)$.
- h) What is the energy eigenvalue for each momentum eigenfunction in your expression from part (g)?
- i) Write down the fully time-dependent wavefunction $\Psi(x,t)$ by tacking-on the appropriate exponential factor to your expression from part (g).
- j) Think about the momentum components that make up $\Psi(x,t=0)$. Qualitatively speaking, what do you expect to happen to the center and width of $\Psi(x,t)$ as time goes forward? (You don't need to evaluate the expression from part (i) to answer this, just explain your reasoning.)