Physics 401 - Homework #2

1) **Expectation value of a discrete variable (six points total).** The "quantum" in quantum mechanics refers to the fact that the energy of bound states is discrete, or quantized, in microscopic systems. Suppose that we have 15 identical quantum systems, and we precisely measure the energy of each one. We find the following results, measured in units of electron-Volts (eV):

$$\{E_i\} = \{8,5,4,5,4,6,7,5,6,4,4,5,6,7,4\}$$

a) Draw a histogram of these results.
b) What was the probability of getting each of the five energy values?
c) What was the most probable value?
d) Calculate the expectation value of the energy of this system.
e) Calculate the expectation value of the square of the energy of this system.
f) Calculate the variance and standard deviation of the energy of this system.

2) **Expectation value of a continuous variable (six points total).** Energy is not always quantized, even in quantum mechanics. For example, free particles can have a continuum of energies. Suppose we have a very large number of identical free particles, and after measuring the energies of all of them, we find that the probability distribution of the energy in electron-Volts is described by

$$P(E) = \left\{ \begin{array}{c} 0, E < 3 \\ \alpha(E-3), 3 \le E \le 8 \\ 0, E > 8 \end{array} \right\}$$

where (α) is a normalization constant.

a) Calculate the normalization constant (α). What are its units?
b) Sketch the normalized probability distribution.
c) What is the probability of measuring an energy between 7.0 eV and 7.1 eV?
d) Calculate the expectation value of the energy of this system.
e) Calculate the expectation value of the square of the energy for this system.
f) Calculate the variance and standard deviation of the energy of this system.

3) **Ortho-normality condition for periodic Fourier Series (three points)**.

Show that

$$\frac{1}{2L}\int_{-L}^{L}\left(e^{in\pi x/L}\right)\left(e^{-im\pi x/L}\right)dx = \delta_{nm} \equiv \begin{cases} 1, n = m \\ 0, n \ne m \end{cases}$$

by explicitly evaluating the integral for both cases. ((n) and (m) are both integers.)

4) **Energy and momentum eigenfunctions (six points total).**

A quantum mechanical particle is in the following state:

$$\psi(x) = A\cos(kx),$$

where (k) is some particular wavenumber whose numerical value is known.

a) Show that this wavefunction is an eigenfunction of the Hamiltonian of a free particle. In other words, show that it satisfies the appropriate eigenvalue equation.

b) What is the corresponding energy eigenvalue for the free particle Hamiltonian?

c) Re-write the wavefunction ψ(x) as a sum of one or more eigenfunctions of the momentum operator.

d) What is/are the eigenvalue(s) of the momentum eigenfunction(s) that appear in your sum in part (c)?

e) Is the wavefunction ψ(x) an eigenfunction of the momentum operator? Why or why not? (You may justify your answer with either a physical argument or by showing the math.)

f) Show that the wavefunction ψ(x) is **not** an eigenfunction of the Hamiltonian if the system contains a potential V(x) which depends on (x).

5) **Non-commuting operators (three points total)**

a) Show that if the momentum operator and position operator are applied to a generic function f(x), the result depends on which operator you apply first. In other words, show that

$$\hat{x}\hat{p}f(x) \neq \hat{p}\hat{x}f(x) .$$

b) Use your work from part (a) to find a simpler expression for this operator:

$$\hat{x}\hat{p} - \hat{p}\hat{x}$$