Physics 401 - Homework #6 - Due Wednesday October 21th

1) **Expectation value of a squared Hermitian operator (three points).** In many respects, Hermitian operators are analogous to a real numbers. For example, we say that an operator is Hermitian if it is equal to its Hermitian conjugate, just as a number is real if it is equal to its complex conjugate. Another similarity between Hermitian operators and real numbers is that the expectation value for a squared Hermitian operator is always greater than or equal to zero, just as the square of a real number is always greater than or equal to zero. Prove this statement for a generic Hermitian operator $\hat{A}$. In other words, show that if $\hat{A} = \hat{A}^t$, then $\left\langle \hat{A}^2 \right\rangle \geq 0$, where the expectation value is evaluated for an arbitrary quantum mechanical state $\left|\psi\right\rangle$. Hint: write down the explicit expression for this expectation value in the position basis, and show that it must be non-negative.

2) **A bracket written "backwards" (one point each)**. We use three types of mathematical objects in quantum mechanics: complex numbers, bra and ket vectors, and operators.

a) If $\left|a\right\rangle$ and $\left|b\right\rangle$ are ket vectors, what type of object is $\left\langle a|b\right\rangle$?

b) Explain in a sentence or two the mathematical effect of $\left\langle a|b\right\rangle$ on an arbitrary ket vector $\left|\psi\right\rangle$.

c) What type of object is $\left|a\right\rangle\left\langle b\right|$?

d) Explain in a sentence or two the mathematical effect of $\left|a\right\rangle\left\langle b\right|$ on an arbitrary ket vector $\left|\psi\right\rangle$.

e) Explain in a sentence or two the mathematical effect of $\left|a\right\rangle\left\langle b\right|$ on an arbitrary bra vector $\left\langle\psi\right|$.

3) **A continuous observable (one point each).** Suppose we have an observable Λ, with associated operator $\hat{\Lambda}$, and which has a continuous set of eigenstates $\left\{\left|\lambda\right\rangle\right\}$. Here we have used the symbol (λ) to label the continuous eigenstates. Just to be confusing, let's also use the symble (λ) to stand for the continuum of eigenvalues for these states.

a) Write down the eigenvalue equation for this observable in Dirac notation.
b) Write down an orthonormality condition for these eigenstates.
c) In one or two sentences, explain how we know that these states must satisfy an orthonormality condition.
d) Write down the mathematical statement of completeness for these states.
e) Expand an arbitrary state $\left|\psi\right\rangle$ in terms of these eigenstates.
f) What is the physical meaning of $\left\langle\lambda|\psi\right\rangle$? (Explain in one or two sentences.)

g) Assuming the usual meaning for the eigenstates $\{|x\rangle\}$, what is the physical meaning of $\langle x|\lambda\rangle$? (Explain in one or two sentences.)

4) **Pythagorean theorem for orthogonal states (two points).** We use the term "norm" to mean the length of a bra or ket vector. In Dirac notation, the squared norm of a state $|\psi\rangle$ is defined by

$$\|\psi\|^2 \equiv \langle\psi|\psi\rangle$$

Show that if two states $|\psi\rangle$ and $|\varphi\rangle$ are orthogonal, then their norms obey the Pythagorean theorem:

$$\|\psi+\varphi\|^2 = \|\psi\|^2 + \|\varphi\|^2$$

Note: The state symbolized by $|\psi+\varphi\rangle$ is meant to be interpreted as the state $|\psi\rangle+|\varphi\rangle$. Strictly speaking, the labels (ψ) and (φ) cannot be mathematically added together, just as you cannot mathematically add your first and last name. So we are playing a dangerous game with our notation by using (ψ+ϕ) as a label. However, people play this game all the time, so you should get used to it.

5) **Promotion and demotion operators.** The "demotion" operator is defined by

$$\hat{a} \equiv \sqrt{\frac{m\omega_0}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega_0}\right)$$

where ω_0 is a frequency relevant to the harmonic oscillator problem.

a) What is the Hermitian conjugate operator, $\hat{a}^t$? (This operator is known as the promotion operator.) (one point)

b) Can the demotion operator stand for an observable? Why or why not? (one point)

c) The commutator of any two operators ($\hat{A}$) and ($\hat{B}$) is defined by

$$[\hat{A},\hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$$

Use the position-momentum commutator to calculate $[\hat{a},\hat{a}^t]$. (three points)