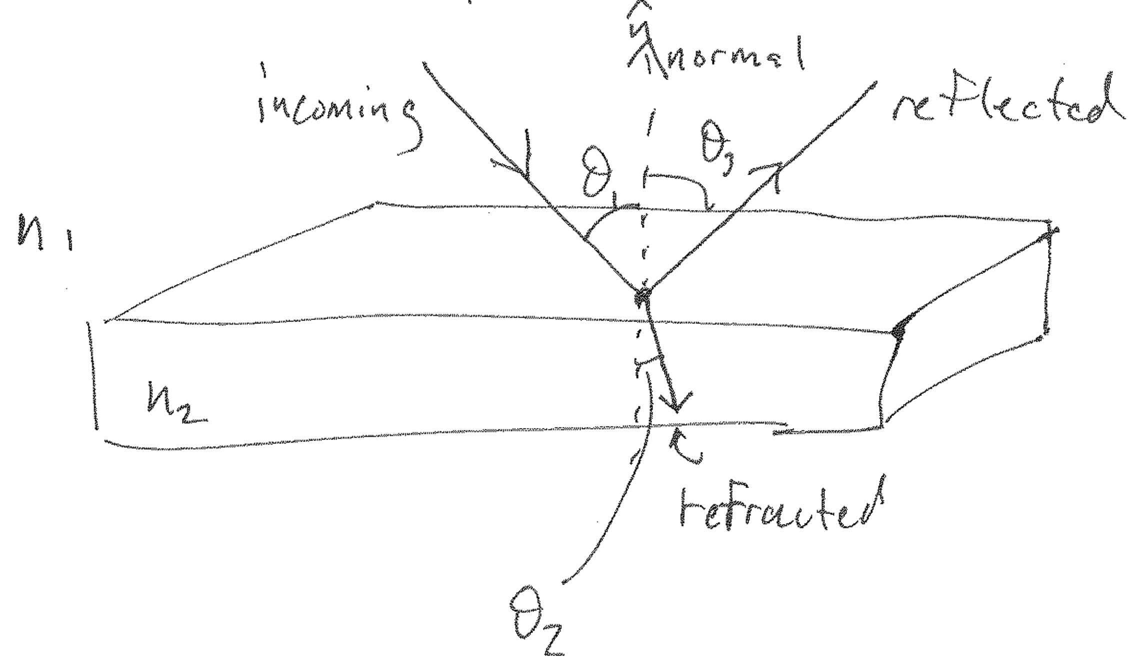
Reflection & Refraction

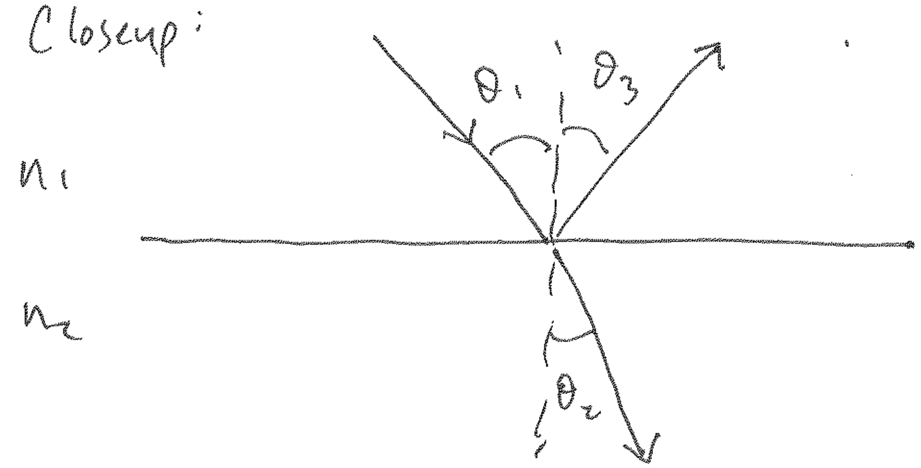
A slab of transparent material:



Snell's Law of Refraction: $n_1 \sin\theta_1 = n_2 \sin\theta_2$

Law of Reflection: $\theta_1 = \theta_3$

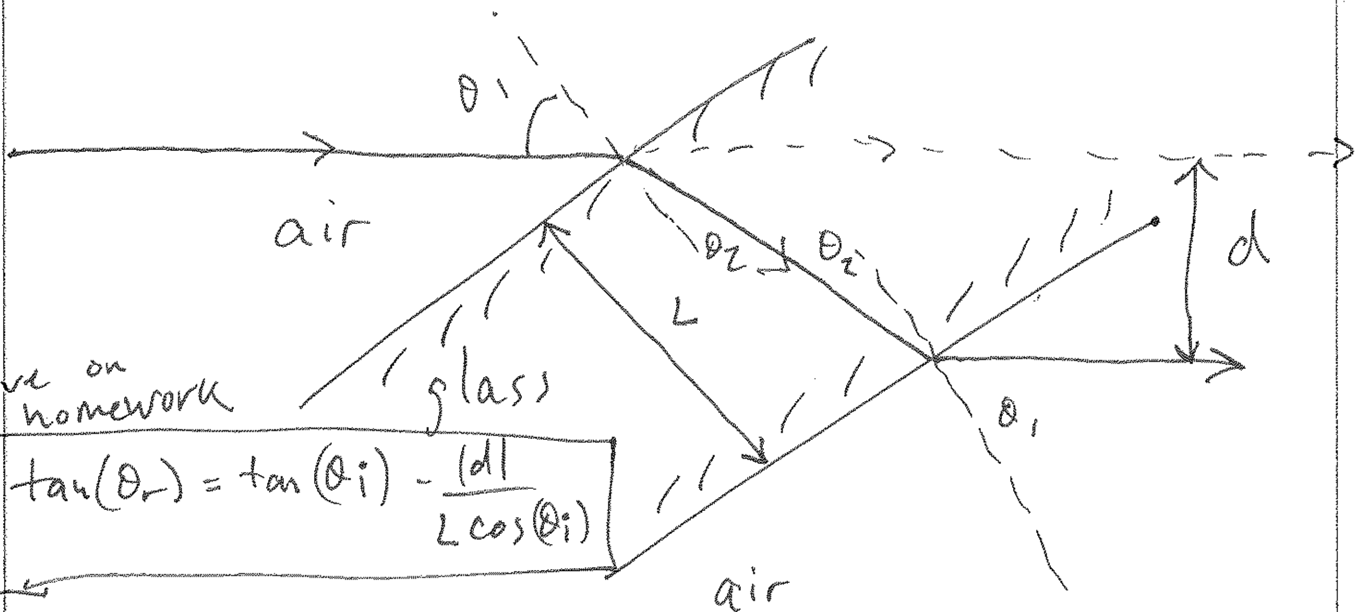
Closeup:



If $n_2 > n_1$, (like air-to-glass) bend towards the normal.

If $n_1 > n_2$, (like glass-to-air) bend away from normal.

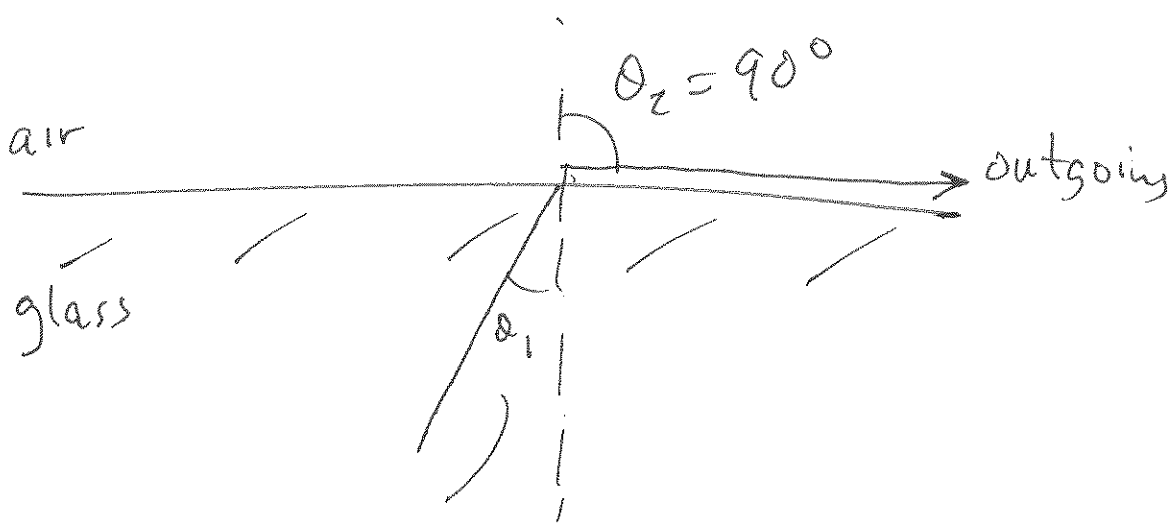
From Homework #2: beam is displaced if it refracts twice through a sheet of glass with parallel sides:



Prove on homework

$$\boxed{\tan(\theta_r) = \tan(\theta_i) - \frac{|d|}{L\cos(\theta_i)}}$$

Total Internal Reflection: If $n_2 > n_1$, like glass-to-air, the refraction angle can be 90°:



we call this angle-of-incidence the "critical angle."

Then $\overset{\text{glass}}{\overset{\downarrow}{n_2}} \sin\theta_c = \overset{\text{air}}{\overset{\downarrow}{n_1}} \sin 90^\circ = n_1$

$$\theta_c = \sin^{-1}\left(\frac{n_1}{n_2}\right)$$

If $n_1 = \text{air} = 1$, then $\boxed{\theta_c = \sin^{-1}\left(\frac{1}{n_2}\right)}$

If the angle of incidence is larger than the critical angle, then there is no transmitted beam. 100% reflection.

Measure θ_c, with uncertainties, and infer n_2 (glass), with uncertainties.

Propagating errors on angles:

Suppose $\theta = 35^\circ \pm 5^\circ$.

Then $\sin\theta = 0.574$. what is the uncertainty on $\sin\theta$?

Propagation of errors formula:

$$f(\theta) = \sin(\theta)$$

Then $$|\sigma_f| = \left|\frac{d(\sin\theta)}{d\theta}\,\sigma_\theta\right| = |(\cos\theta) \times \sigma_\theta|$$
$$= |\cos(35^\circ) \times \sigma_\theta|$$
$$= (0.819) \times \sigma_\theta$$

What should we use for σ_θ?

Wrong answer: $\sigma_\theta = 5°$. Lets see what happens.

???

???

$$|\sigma_F| = (0.819) \times (5°) = 4.10 \text{ ???}$$

wrong!

so $\boxed{\sin\theta \overset{?}{=} 0.574 \pm \underbrace{4.10}_{}}$???

way too large!

sin θ should be less than one.

Here's the problem:

$$\underbrace{\sigma_{\sin\theta}}_{\substack{\text{should}\\ \text{be}\\ \text{unitless}}} = \sigma_F = \overset{\cos(35°)}{\underbrace{(0.819)}_{\substack{\cos\theta\\ \text{is}\\ \text{unitless}}}} \times \underbrace{(5°)}_{}$$

this has units! wrong!!

Right answer: Need σ_θ to be unitless, so use radians. Then $\sigma_\theta = \left(\frac{5°}{180°}\right)\pi = 0.087$

Then $\sigma_F = \cos\theta\,\sigma_\theta = (0.819)(0.087 \text{ radians})$

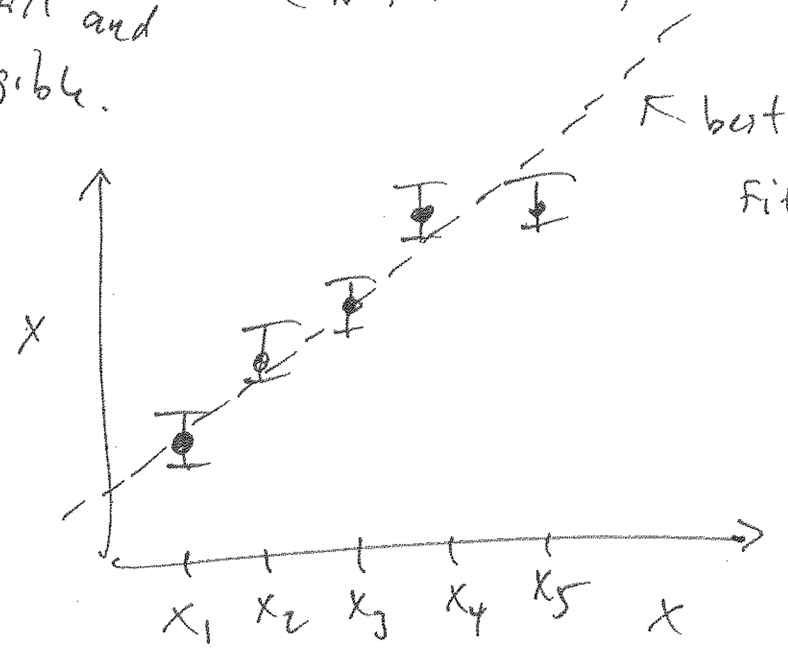
correct $\boxed{\sigma_F = 0.071}$, so $\boxed{\sin\theta = 0.574 \pm 0.071}$

Linear Least-squares Fits.

A set of data: $(x_1, y_1 \pm \sigma_{y_1})$,
$(x_2, y_2 \pm \sigma_{y_2})$,
$(x_3, y_3 \pm \sigma_{y_3})$,
$\vdots$
$(x_N, y_N \pm \sigma_{y_N})$

uncertainties on x_i are small and negligible.

Plot it:



Hypothesis: Linear Relationship: $y = a + bx$
What are the best values for a & b??

Answer: The χ^2 is $\chi^2 = \sum_{i=1}^{N} \left[\frac{(a + bx_i) - y_i}{\sigma_{y_i}} \right]^2$

uncertainty of y value $\{i\}$
(σ_{y_i})

$$\chi^2 = \left(\frac{(a+bx_1)-y_1}{\sigma_{y_1}}\right)^2 + \left(\frac{(a+bx_2)-y_2}{\sigma_{y_2}}\right)^2 + \ldots\ldots$$

What are the best values for a & b?

Answer. Define

$$\boxed{\Delta \equiv \sum_i \frac{1}{\sigma_i^2} \sum \frac{x_i^2}{\sigma_i^2} - \left(\sum \frac{x_i}{\sigma_i^2}\right)^2}$$

Then

$$\text{best } a = \frac{1}{\Delta}\left[\left(\sum_i \frac{x_i^2}{\sigma_i^2}\right)\left(\sum_i \frac{x_i}{\sigma_i^2}\right) - \left(\sum_i \frac{x_i}{\sigma_i^2}\right)\left(\sum_i \frac{x_i y_i}{\sigma_i^2}\right)\right]$$

$$\text{best } b = \frac{1}{\Delta}\left[\left(\sum_i \frac{1}{\sigma_i^2}\right)\left(\sum_i \frac{x_i y_i}{\sigma_i^2}\right) - \left(\sum_i \frac{x_i}{\sigma_i^2}\right)\left(\sum_i \frac{y_i}{\sigma_i^2}\right)\right]$$

Crucially, what are the uncertainties on (a) & (b)?

Answer:

$$\boxed{\sigma_a = \sqrt{\frac{1}{\Delta}\left(\sum_i \frac{x_i^2}{\sigma_i^2}\right)}}$$

$$\boxed{\sigma_b = \sqrt{\frac{1}{\Delta}\left(\sum_i \frac{1}{\sigma_i^2}\right)}}$$

See also:

Taylor, 2nd Edition, Problems 8.9 & 8.19

Bevington, 3rd Edition, Equations 6.12, 6.21, & 6.22

Lyons, Equations 2.10, 2.13', 2.16'