Phys 375 – Homework #2

1) (6 pts) Snell's law says $n_1 \sin(\theta_1) = n_2 \sin(\theta_2)$. This function is non-linear in (θ_1) and (θ_2), but is linear in $\sin(\theta_1)$ and $\sin(\theta_2)$, and therefore you can do a linear straight-line χ^2 fit to determine (n_2) if (n_1) is known. You measure the refraction of a laser beam which starts in air (n = 1) and transmits into glass. After taking the sine of your angular measurements, you have the following data:

sin(θ1)	sin(θ2)
-0.500	-0.310
-0.342	-0.173
-0.174	-0.204
0.000	0.091
0.174	0.115
0.342	0.197
0.500	0.378

You estimate that the error on each of the $\sin(\theta_2)$ measurements is 0.05. The uncertainty in $\sin(\theta_1)$ is negligible.

a) Do a single-parameter straight-line fit to determine (n_2) and its uncertainty from this data. Report the reduced χ^2 of the fit.

b) Turn in a plot of $\sin(\theta_2)$ vs $\sin(\theta_1)$, showing the error bars on each point, and the best-fit linear function.

2) *(9 pts) Determining the minimum in a quadratic.*

1) Consider the following quadratic function:

$$y(x) = a + bx + cx^2$$

Show that x_{min} = -b/(2c), where x_{min} is the x-value that minimizes the function.

2) Suppose that (a), (b), and (c) have been determined to have the following values and uncertainties: $a = a_0 \pm \sigma_a$, $b = b_0 \pm \sigma_b$, and $c = c_0 \pm \sigma_c$. Show that the propagation-of-errors formula implies that the error on x_{min} is

$$\sigma_{x\min} = \sqrt{\left(\frac{\sigma_b}{2c_0}\right)^2 + \left(\frac{b_0\sigma_c}{2c_0^2}\right)^2}$$

3) Suppose that b = 20.5±2.0 and c = 0.80±0.08. Calculate the uncertainty on xmin by using the propagation of errors formula. Cross-check the formula by altering, one at a time, (b) by its uncertainty, and (c) by its uncertainty, and observing the resulting changes in x_{min}. Is the propagation-of-errors formula exactly correct for the uncertainty in x_{min} due to (b)? Is it exactly correct for the uncertainty in x_{min} due to (c)? Why or why not, in both cases?

3) (9 pts) Consider the following set of data x-y:

x	y
13	33.02
14	28.54
15	20.82
16	24.80
17	22.64
18	24.32
19	31.15
20	38.60
21	43.59

The x-values are very precise, so they have no uncertainty, but the y-values have an estimated uncertainty of ±1.6.

1) Do a χ^2 fit to a quadratic function $y(x) = a + bx + cx^2$ and determine the best-fit values on the parameters (a), (b), and (c). Also determine the uncertainties on those parameters by taking into account the uncertainty on the y-values. Report the reduced χ^2 of the fit.

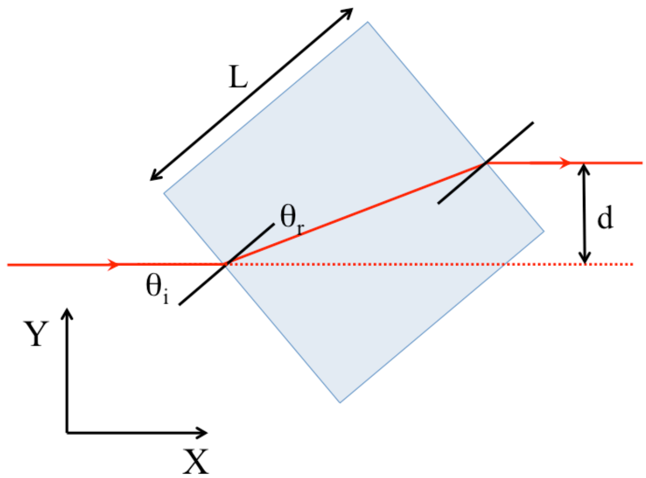
    a. Comment: Even though the fit function is non-linear in (x), nevertheless this is a linear fit, because the function is linear in the fit parameters (a), (b), and (c). Because this is a linear fit, the best-fit values and their uncertainties can be determined analytically. See, for example, the textbooks by Taylor, or Lyons, or Bevington.
    b. Hint: You can use a computer program like excel, Mathematica, or Matlab to determine the best-fit parameters and their uncertainties. See, for example, youtube video "Fitting data in Mathematica" by Video Tutorials (PER@C), or the `fitlm` function in Matlab (http://www.mathworks.com/help/stats/fitlm.html). You can also find some online tools for linear fits with uncertainties. See, for example, http://www.physics.csbsju.edu/stats/WAPP.html

2) Turn-in a plot of the data showing the error bars and the best-fit quadratic function.

3) Use your fitted values for the parameters and their uncertainties to determine x_{min}, the x-value that minimizes the quadratic function, and its uncertainty. You may use the formula from Question 2 part (2) above.

Comment: the formula from Question (2) part (2) actually over-estimates the uncertainty on x_{min} for this data. This is because it neglects correlations in the uncertainties between parameters (b) and (c), and these correlations are important in this case. We will learn *about how to handle correlations later in the course. For now you may use the formula as given.*

4) (15 pts) A ray of light is traveling in the x-direction. It meets a rectangular sheet of glass of thickness (L) that is tilted so that the ray's angle of incidence is (θ_i). Let the angle of refraction be called (θ_r).



a) Using Snell's law, show that the ray emerges from the sheet of glass traveling parallel to the x-axis.

b) Show that the ray has been displaced in the y-direction by an amount (d) with magnitude

$$|d| = \frac{L\sin(\theta_i - \theta_r)}{\cos\theta_r}$$

c) Re-write this expression as

$$\tan(\theta_r) = \tan(\theta_i) - \frac{|d|}{L\cos(\theta_i)}$$

d) You measure $|d| = 4.3\pm0.1$ mm, $L = 10.8\pm0.1$ mm, and $\theta_i = 50.2\pm0.1$ degrees. Use the formula from part (c) to determine the central value of the index of refraction (n) for this data.

e) Separately determine the uncertainty on the index of refraction (n) due to the uncertainty on each input measurement, one at a time: (d), (L), and (θ_i).

Hint 1: The propagation-of-errors formula for the equation from part (c) is very long and tedious to evaluate. It is much simpler to determine these uncertainties with a spreadsheet or some other computer program. Use the spreadsheet to evaluate (n) with the help of the equation from part (c). Then you can easily vary the input values one at a time and observe the resulting change in the index of refraction.

Hint 2: Make sure that you are giving the angles to the trigonometric functions in your spreadsheet in the units that they expect (either degrees or radians).

f) Combine together the uncertainties from part (e) to get the total uncertainty on (n). Which uncertainty contributes the most? *Hint: The total relative uncertainty should be no more that a few percent.*

5) (6 pts) You measure the critical angle for total internal reflection in a piece of glass to be 43.5 ± 1.0 degrees. Use two methods to show that the uncertainty on the index of refraction is 0.026.

a) Method 1: 'experimentally' determine the error on (n) by varying the critical angle by its uncertainty and observing the change in the (n).

b) Method 2: use the formal propagation-of-error formula.

Hint: If you are having trouble showing that the uncertainty on (n) is 0.026 with the propagation-of-errors formula (method 2), make sure you are converting the uncertainty in the critical angle to radians before plugging into the formula. Why should you use radians and not degrees? Because both (n) and $\sin(\theta_c)$ *are unitless quantities, and the uncertainty on* θ_c *must also unitless for the propagation-of-errors formula to make sense. If you fail to convert to radians, your calculated uncertainty will be too large by a factor of 57.3, which is the number of degrees in one radian.*