Correlated Errors

Once per day I measure the length and width of a rectangular peice of quartz using a metal ruler. I repeat each day for many weeks. From the measuremnts I calculate the area of the quartz:

A = length × width = $l \times w$

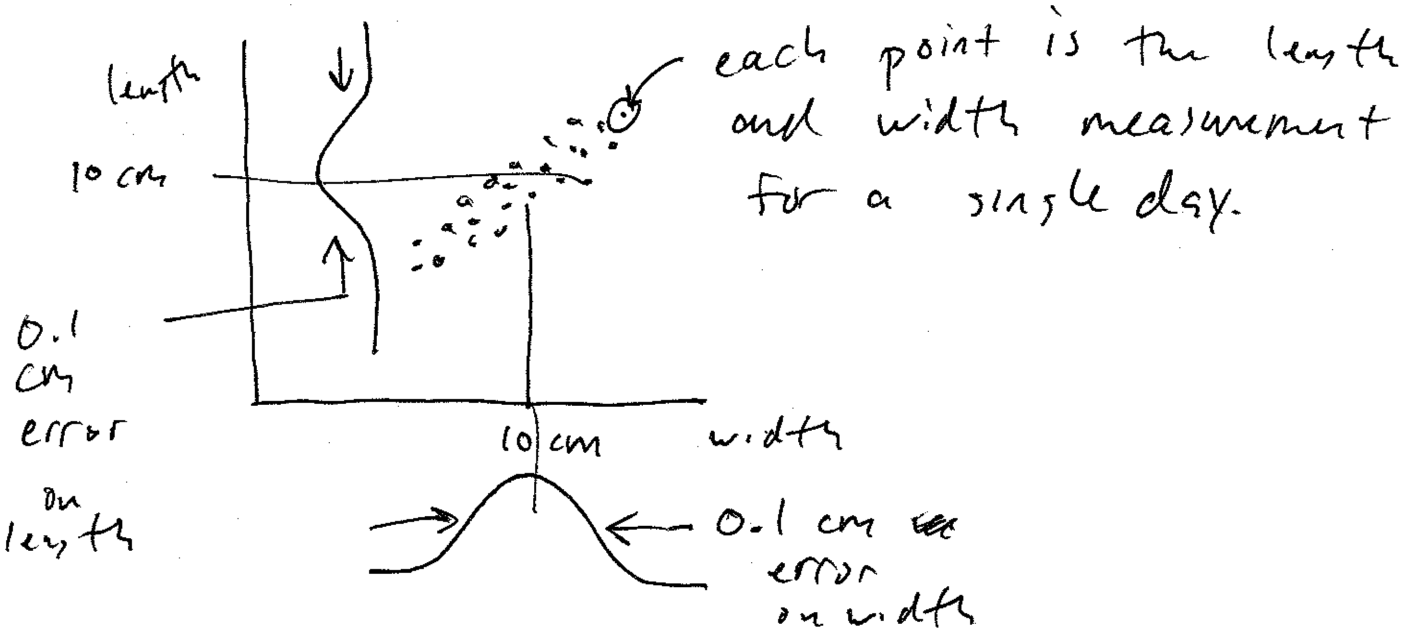
Uncertainty of Area: $$\left(\frac{\sigma_A}{A}\right)^2 = \underbrace{\left(\frac{\sigma_l}{l}\right)^2 + \left(\frac{\sigma_w}{w}\right)^2}$$

Follows from the propagation of errors formula.

This is correct as long as the errors on l are unrelated (uncorrelated) to the errors on w. For example, if the dominant error on l and w is ~~the~~ my ability to correctly read the ruler ~~of them~~ (a completely random error), Then the (w) errors will be uncorrelated with the (l) errors.

However, suppose that the most important error arises from the thermal expansion and contraction of the ruler. Each day the temperature changes a little bit. One day the ruler is a bit short, and then both (ℓ) and (w) appear a bit larger than average. The next day the ruler is a bit long, and both (ℓ) and (w) appear a bit smaller than average.

The data look like:



Naive propagation of errors gives

$$\left(\frac{\sigma_A}{A}\right)^2 = \left(\frac{\sigma_w}{w}\right)^2 + \left(\frac{\sigma_\ell}{\ell}\right)^2$$

$$= \left(\frac{0.1}{10}\right)^2 + \left(\frac{0.1}{10}\right)^2$$

$$= 2\,(0.01)^2$$

Therefore $A = 100 \text{ cm}^2$

and $\frac{\sigma_A}{A} = (\sqrt{2})(0.01)$

or $\sigma_A = (\sqrt{2})(0.01)(100 \text{ cm}^2)$

$\sigma_A = 1.4 \text{ cm}^2$ ← However this result is wrong because we have not accounted for the correlation btw (l) & (w) errors.

In fact, if we take 100 measurements and calculate A for each and histogram the results,

Propagation of Errors with Correlations:

Area = $A(l,w) = l \times w$

Assume a series of measurements
$\{A_i\} = \{l_i w_i\}$, N measure

$$\sigma_A^2 = \lim_{N \to \infty} \left[\frac{1}{N} \sum_i (A_i - \bar{A})^2 \right]$$

← mean value

But A_i depends on l_i & w_i:

$$A_i - \bar{A} \approx \frac{\partial A}{\partial w}(w_i - \bar{w}) + \frac{\partial A}{\partial l}(l_i - \bar{l})$$

Thus $$\sigma_A^2 \approx \lim_{N \to \infty} \frac{1}{N} \sum_i \left[\frac{\partial A}{\partial w}(w_i - \bar{w}) + \frac{\partial A}{\partial l}(l_i - \bar{l}) \right]^2$$

$$= \lim_{N \to \infty} \frac{1}{N} \sum_i \Bigg[\underbrace{\left(\frac{\partial A}{\partial w}\right)^2 (w_i - \bar{w})^2 + \left(\frac{\partial A}{\partial l}\right)^2 (l_i - \bar{l})^2}_{\text{the usual terms, independent of correlations}}$$

$$+ \underbrace{2\left(\frac{\partial A}{\partial w}\right)\left(\frac{\partial A}{\partial l}\right)(w_i - \bar{w})(l_i - \bar{l})}_{\text{correlation term.}} \Bigg]$$

we define $$\boxed{\sigma_{lw}^2 \equiv \frac{1}{N} \sum_i (w_i - \bar{w})(l_i - \bar{l})}$$

"The covariance of l & w"

Also define $$\sigma_l^2 \equiv \frac{1}{N} \sum (\bar{l} - l_i)^2$$ "variance of l"

$$\sigma_w^2 \equiv \frac{1}{N} \sum (\bar{w} - w_i)^2$$ "variance of w"

Then
$$\sigma_A^2 = \sigma_w^2 \left(\frac{\partial A}{\partial w}\right)^2 + \sigma_l^2 \left(\frac{\partial A}{\partial l}\right)^2 + \underbrace{2\,\sigma_{lw}^2 \left(\frac{\partial A}{\partial l}\frac{\partial A}{\partial w}\right)}_{\text{new term due to correlation.}}$$

For our example:

$$\sigma_W^2 = (0.1\ cm)^2 = 0.01\ cm^2$$

$$\sigma_l^2 = (0.1\ cm)^2 = 0.01\ cm^2$$

$$\sigma_{lw}^2 \approx \emptyset$$ ← for uncorrelated errors.

$$\sigma_{lw}^2 \approx (0.1\ cm)^2$$
$$= 0.01\ cm^2$$
← For 100% correlated

$$\sigma_{lw}^2 = -(0.1)^2$$ ← 100% anticorrelation

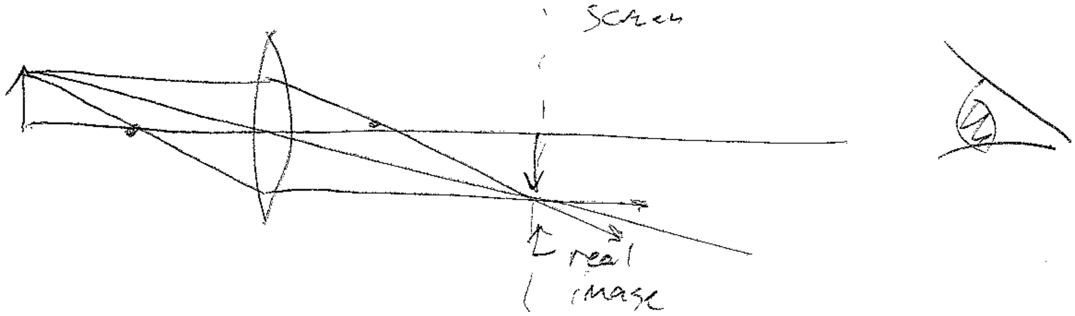
Uncorrelated:

$$\sigma_A^2 = (0.01) + (0.01) + 2(0.01$$

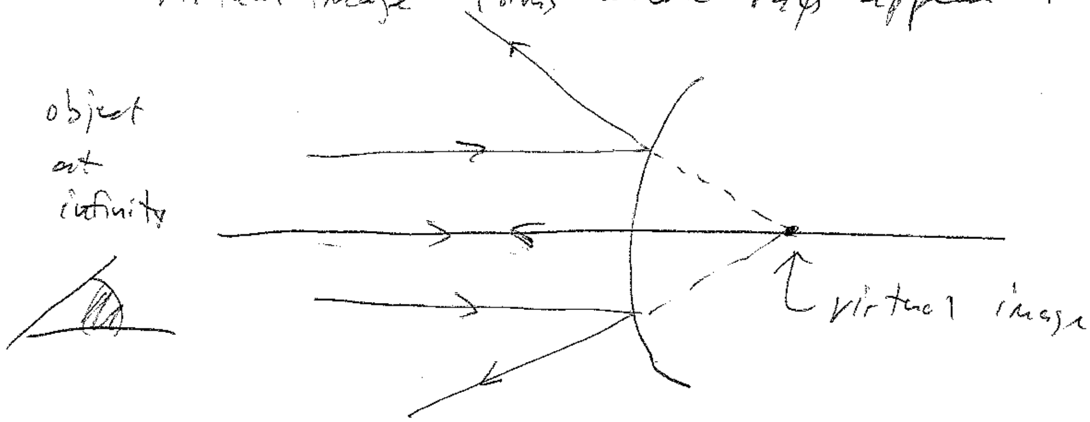
Phys 375 Lecture 4 9/25/07

Recap

- A real image forms when multiple rays converge



- Virtual image forms where rays appear to converge



Spherical Mirror:

$$\frac{1}{s} + \frac{1}{s'} = +\frac{2}{R} \equiv \frac{1}{f}, \quad f = +\frac{R}{2}$$

$s > 0$ object is real

$s' > 0$ image is real

$R > 0$ mirror ~~convex~~ concave

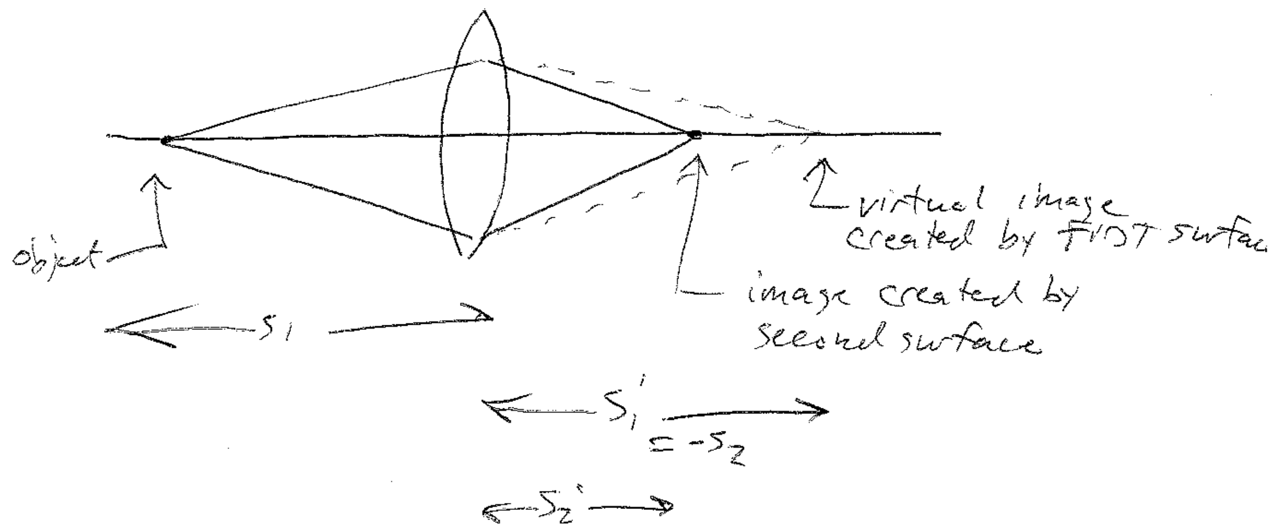
Refraction at curved surface

$$\frac{n_1}{s} + \frac{n_2}{s'} = \frac{n_2 - n_1}{R}$$

Same sign convention

Thin Lenses

Two refractions:



1st refraction: $\dfrac{n_1}{s_1} + \dfrac{n_2}{s_1'} = \dfrac{n_2 - n_1}{R_1}$ (1) $\quad R_1 > 0,\ s_1 > 0,\ s_2 > 0$

2nd refraction: $\dfrac{n_2}{s_2} + \dfrac{n_1}{s_2'} = \dfrac{n_1 - n_2}{R_2}$, $\quad R_2 < 0,\ s_2 = -s_1,\ s_2' > 0$

$$\frac{n_2}{-s_1} + \frac{n_1}{s_2'} = \frac{n_1 - n_2}{R_2} \quad (2)$$

(1) + (2):

$$\frac{n_1}{s_1} + \frac{n_1}{s_2'} = (n_1 - n_2)\left(\frac{1}{R_2} - \frac{1}{R_1}\right)$$

$s \equiv s_1$

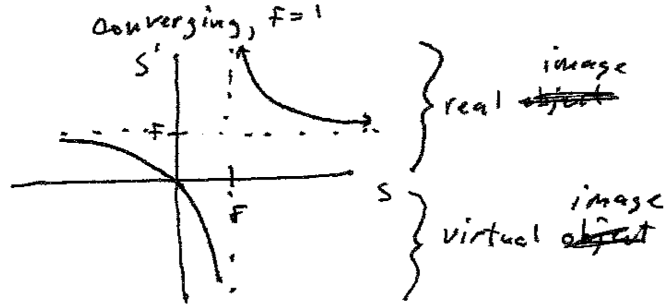
$s' \equiv s_2'$

$$\boxed{\frac{1}{s} + \frac{1}{s'} = \left(\frac{n_2 - n_1}{n_1}\right)\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \text{ Lens Makers Eq}}$$

Define $\frac{1}{F} \equiv \left(\frac{n_2-n_1}{n_1}\right)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$ focal length

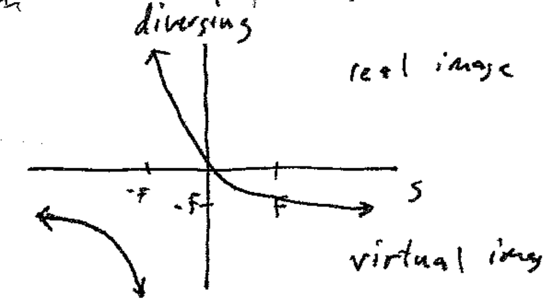
$$\boxed{\frac{1}{s}+\frac{1}{s'}=\frac{1}{F}}$$ thin lens eq.

1) $s>0$ object real
2) $s'>0$ image real
3) $f>0$ converging lens
$f<0$ diverging lens.

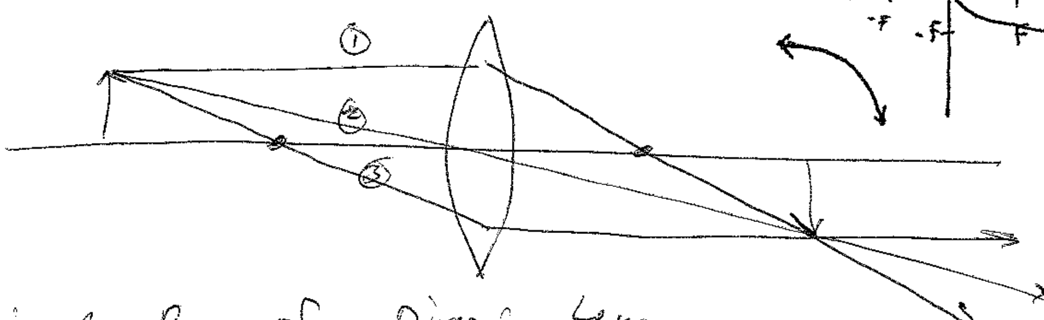


$\frac{1}{F}$ = power, measured in diopters = $\frac{1}{m}$.

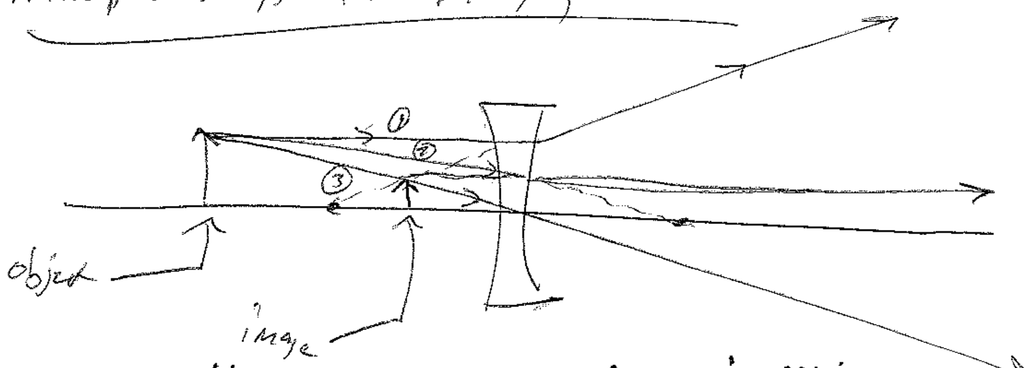
If $F = 20$ cm, $\frac{1}{F} = \frac{1}{0.2\,m} = 5$ diopters



Principle Rays for converging lens:



Principle Rays for Diverging Lens



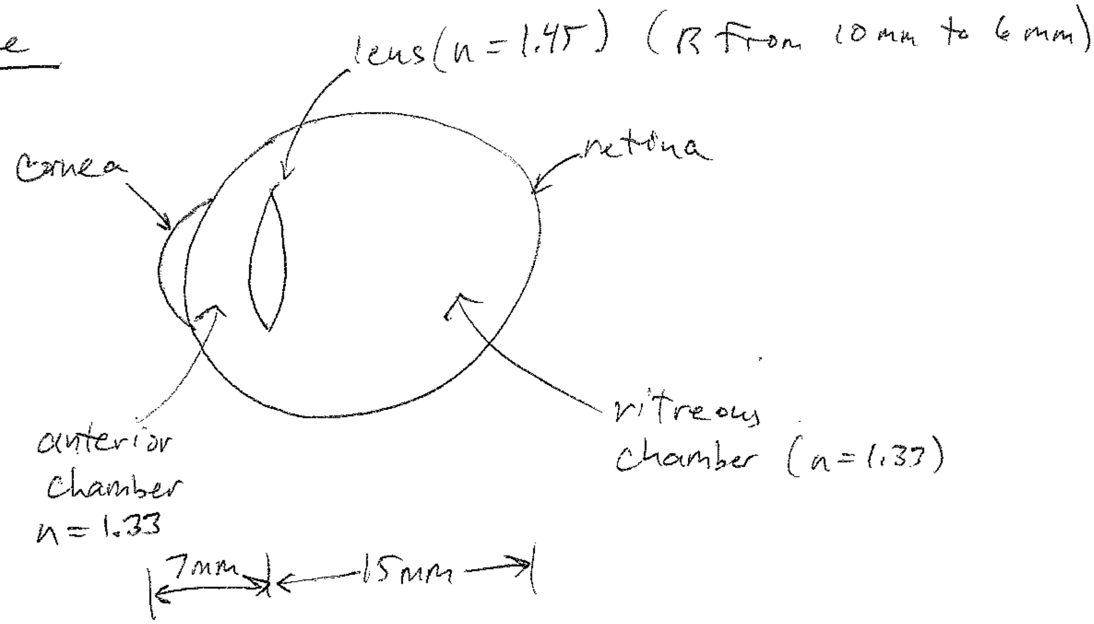
diverging case

real ob → virtual image
virtual ob → real image
virtual ob → virtual image
$\boxed{\text{real ob} \to \text{~~real~~ image}}$

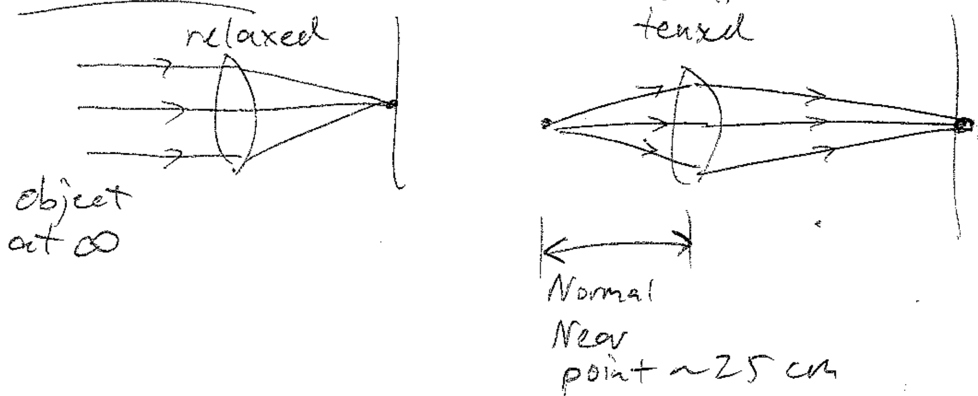
converging case:

real ob → real image
real ob → virtual ~~image~~ image
virtual ob → real image
$\boxed{\text{Virtual ob} \to \text{~~virtual im~~}}$

The eye



Normal Eye



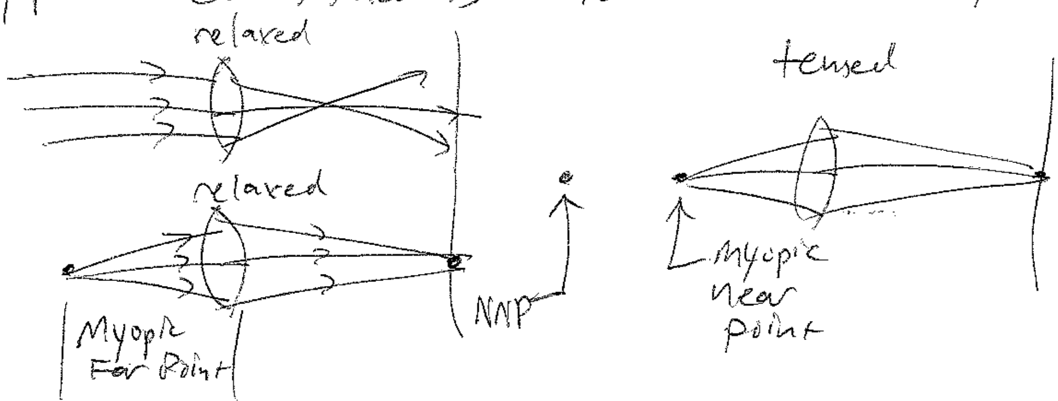
NNP children ~ 7 to 10 cm

middle age ~ 20 to 40 cm } presbyopia

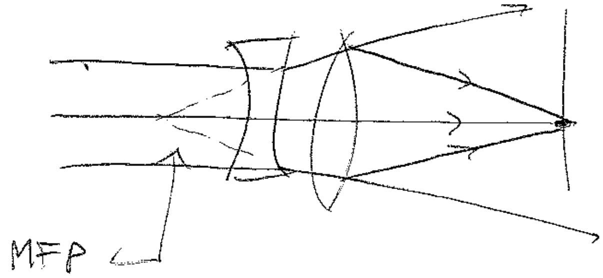
old age ~ 200 cm

Reading glasses put a virtual image at the eye's NNP.

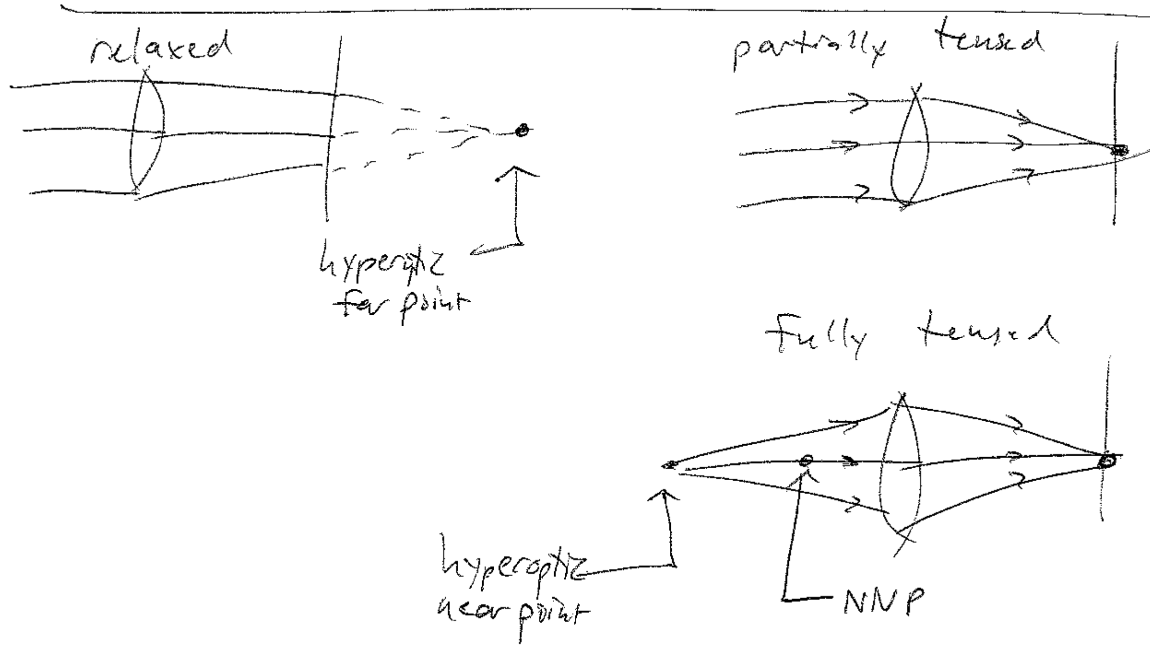
Myopia – Near sightedness – Retina too far away



Myopic Correction - Diverging lens



Hyperoptic Eye - Retina too close - Far Sightedness



Converging Lens Corrects