

Error Analysis

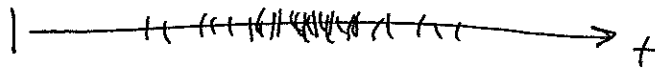
- Sources of errors:
- limitations of apparatus
  - human error
  - finite measurement time
  - electronic noise

Two Types:

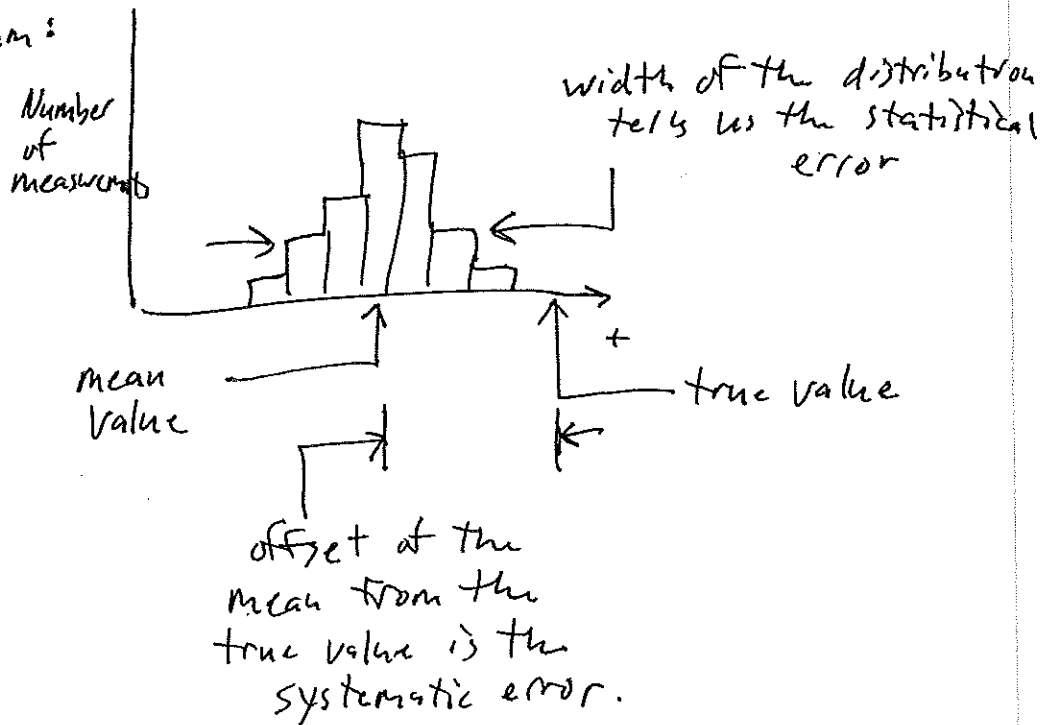
Statistical - errors which vary randomly on every measurement

Systematic - errors which bias the measurement  
 ex: a stopwatch that runs fast always overestimates time

Suppose we measure the period of a pendulum many times:



Make a histogram:



Evaluate statistical errors by repeating the measurement.  
 Evaluate systematic errors by thinking carefully about the apparatus.

A set of measurements  $\{x_i\}$

$$\text{Mean } x = \bar{x} = \frac{\sum_i x_i}{N}$$

Median  $x = \bar{x}_{\frac{1}{2}} \rightarrow$  a value such that one half of the  $x_i$  are above and one half below.

Standard Deviation = statistical error = Root-Mean-Square Dev.

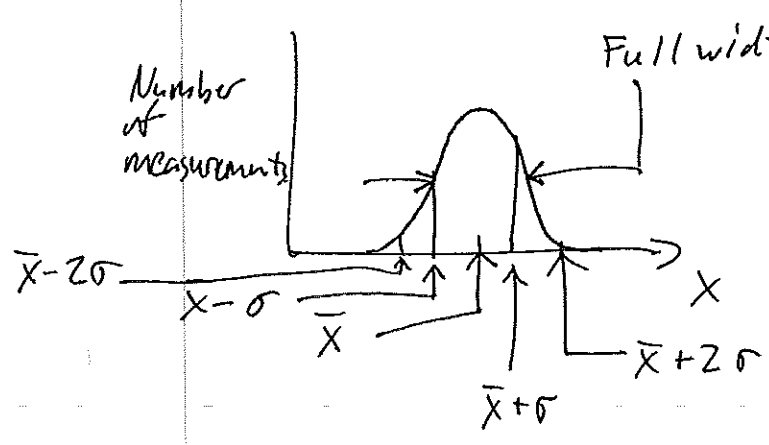
$$s = \sqrt{\frac{\sum_i (x_i - \mu)^2}{N}} \quad \mu = \text{true value of } x, \text{ unknown to the experimenter.}$$

Variance  $\equiv s^2$

$$\sigma = \sqrt{\frac{\sum_i (x_i - \bar{x})^2}{N-1}} \quad \text{mean value of } x, \text{ can be calculated.}$$

the usual way to evaluate a statistical error.

If the errors are truly random, then the histograms will appear gaussian:



- Full width at half maximum =  $2.35 \sigma$  (FWHM)
- 68% of measurements are within  $\pm 1 \sigma$  from  $\bar{x}$ .
- 95% within  $\pm 2 \sigma$
- 99.7% within  $\pm 3 \sigma$

### Standard Deviation of the Mean:

What's the error on  $\bar{x}$ ? Answer:

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{N}}$$

Typically we report a measurement and its statistical error as

$$\bar{x} + \sigma_{\bar{x}} + \text{systematic error}$$

↑ statistical error

### Propagation of Errors

Suppose we use a measurement to calculate another quantity:

$$U = mgh \quad , \quad m = 0.519 \text{ kg}$$

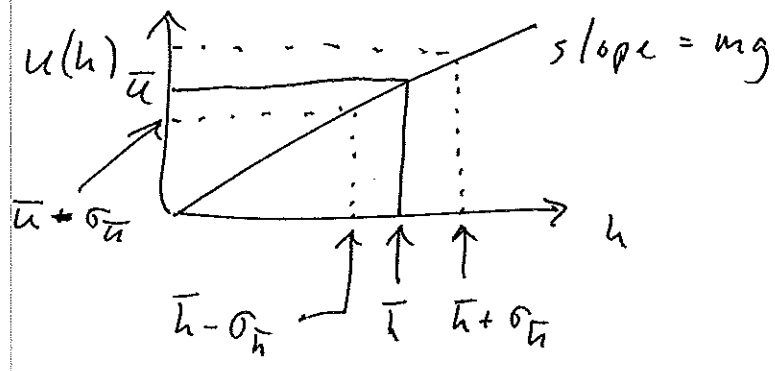
$$g = 9.8 \text{ m/s}^2$$

$$h = 1.0 \pm 0.1 \text{ cm}$$

Central value of  $U$  determined by plugging in:

$$\bar{U} = (0.519 \text{ kg})(9.8 \text{ m/s}^2)(1.0 \times 10^{-2} \text{ m}) = 0.0508 \text{ J}$$

Error on  $U$  due to error on  $h$ ?



$$\frac{\Delta U}{\Delta h} = \frac{dU}{dh} = mg$$

$$\Delta U = mg \Delta h$$

Now Suppose we have errors on all three quantities. Then we use partial derivatives and add the errors in quadrature

$$\Delta U = \sqrt{\left(\frac{\partial U}{\partial h} \Delta h\right)^2 + \left(\frac{\partial U}{\partial m} \Delta m\right)^2 + \left(\frac{\partial U}{\partial g} \Delta g\right)^2}$$

In general, if  $F = F(x, y, z, \dots)$ , then

$$\Delta F = \sqrt{\left(\frac{\partial F}{\partial x} \Delta x\right)^2 + \left(\frac{\partial F}{\partial y} \Delta y\right)^2 + \left(\frac{\partial F}{\partial z} \Delta z\right)^2}$$

$\Delta F$  = absolute error, has same units as  $F$

$\frac{\Delta F}{F}$  = fractional error, unitless.

Two simple cases:

If  $F = a + b + c$ , then  $\Delta F = \sqrt{\Delta a^2 + \Delta b^2 + \Delta c^2}$

If  $F = abc$ , then  $\left(\frac{\Delta F}{F}\right) = \sqrt{\left(\frac{\Delta a}{a}\right)^2 + \left(\frac{\Delta b}{b}\right)^2 + \left(\frac{\Delta c}{c}\right)^2}$

### Waves

Typical Harmonic Wave:  $\psi(x, t) = A \cos(kx - \omega t)$

$A$  = amplitude

$k$  = wave number =  $\frac{2\pi}{\lambda}$ ,  $\lambda$  = wavelength

$x$  = position

$\omega$  = angular frequency =  $\frac{2\pi}{T}$ ,  $T$  = period

=  $2\pi\nu$ ,  $\nu = \frac{1}{T}$  = frequency

$t$  = time.

~~wave velocity~~

Wave velocity : phase is constant

$$kx - \omega t = \text{constant}$$

$$k \frac{dx}{dt} - \omega = 0$$

$$\frac{dx}{dt} = \frac{\omega}{k} = v = \frac{\lambda}{T} = \lambda v$$

### Complex numbers

$$z = a + ib, \quad i = \sqrt{-1}$$

$$\text{Re}(z) = a$$

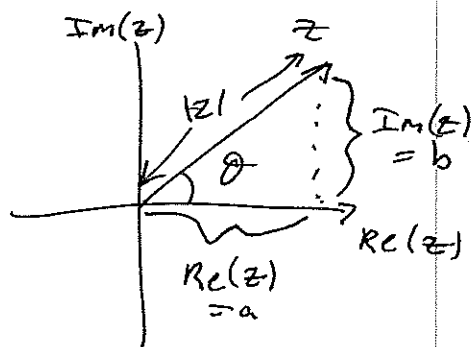
$$\text{Im}(z) = b$$

Magnitude of  $z$ :  $|z| = \sqrt{a^2 + b^2}$

$$\text{Re}(z) = |z| \cos \theta = a$$

$$\text{Im}(z) = |z| \sin \theta = b$$

$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$



$$z = |z| (\cos \theta + i \sin \theta) = |z| e^{i\theta}$$

$e^{i\theta}$  by Euler's formula

Two ways to express a complex number:

$$z = a + ib, \quad \text{or} \quad z = |z| e^{i\theta}, \quad \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

Complex conjugate:  $z^* = a - ib$  or  $|z| e^{-i\theta}$ .

$$z z^* = a^2 + b^2 = |z|^2$$

Write the harmonic wave in complex form:

$$\psi(x,t) = A \cos(kx - \omega t) = \text{Re} \left[ A e^{i(kx - \omega t)} \right]$$

$$\sim A e^{i(kx - \omega t)}$$

remember to take the real part