Phys 375 Fall 07 Lecture 1 9/4/07 1

Error Analysis

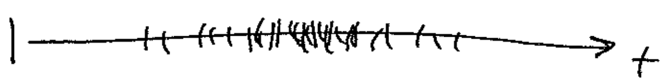
Sources of errors:
- limitations of apparatus
- human error
- finite measurement time
- electronic noise

Two Types:

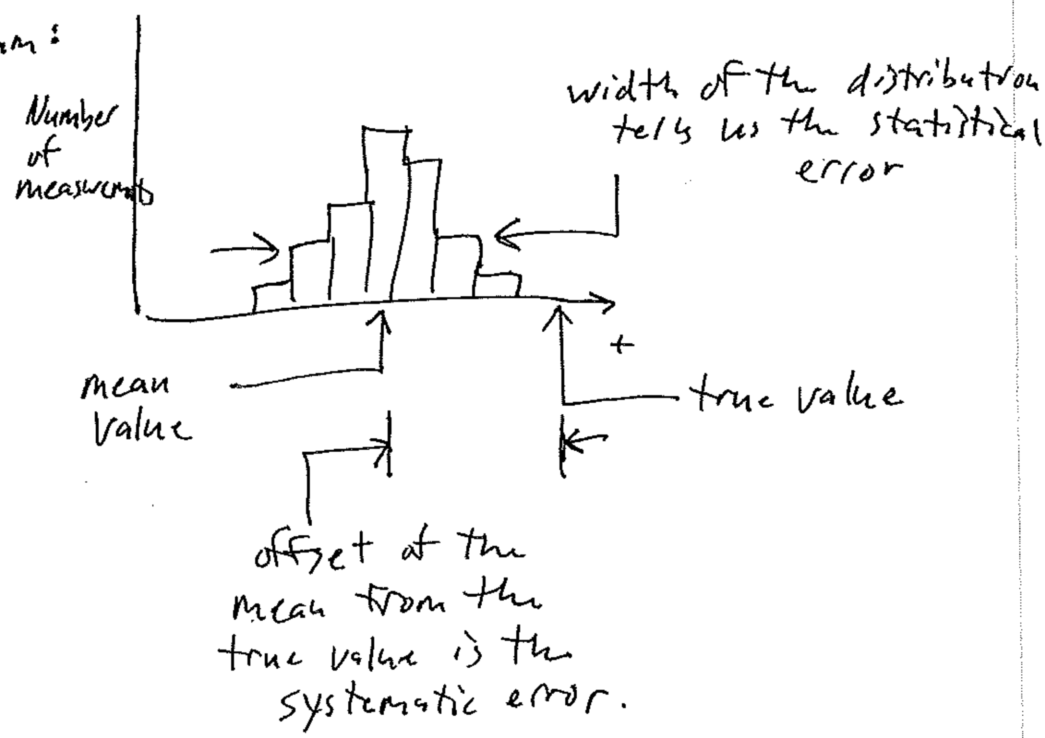
Statistical – errors which vary randomly on every measurement

Systematic – errors which bias the measurement
ex: a stopwatch that runs fast always overestimates time

Suppose we measure the period of a pendulum many times:



Make a histogram:



Evaluate statistical errors by repeating the measurement.
Evaluate systematic errors by <u>thinking carefully</u> about the apparatus.

A set of measurements $\{x_i\}$

Mean $x = \bar{x} = \dfrac{\sum_i x_i}{N}$

Median $x = \bar{x}_{\frac{1}{2}} \rightarrow$ a value such that one half of the x_i are above and one half below.

Standard Deviation = statistical error = Root-Mean-Square Dev.

$$s = \sqrt{\frac{\sum_i (x_i - \mu)^2}{N}}$$

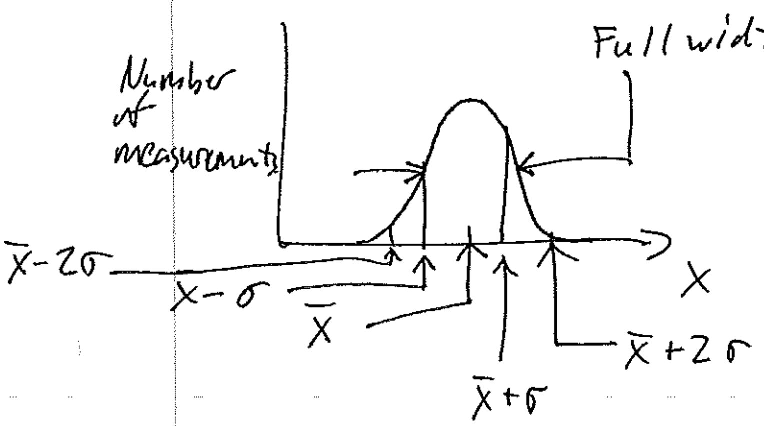
μ = true value of x, unknown to the experimenter

variance $\equiv s^2$

$$\underbrace{\sigma = \sqrt{\frac{\sum_i (x_i - \bar{x})^2}{N-1}}}_{\text{the usual way to evaluate statistical error.}}$$

$\bar{x}$ = mean value of x, can be calculated.

If the errors are truly random, then the histogram will appear gaussian:



Full width at half maximum = 2.35 σ (FWHM)

68% of measurements are within ±1σ from $\bar{x}$.

95% within ±2σ

99.7% within ±3σ

Standard Deviation of the Mean:

What's the error on $\bar{x}$? Answer: $\boxed{\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{N}}}$

Typically we report a measurement and its statistical error as

$$\bar{x} + \underbrace{\sigma_{\bar{x}}}_{\text{statistical error}} + \text{systematic error}$$

Propagation of Errors

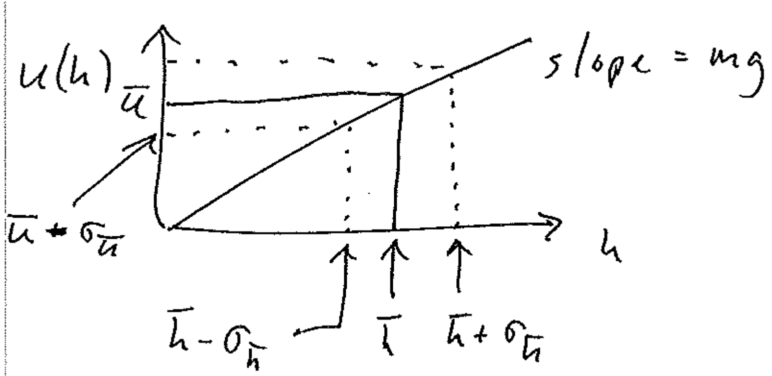
Suppose we use a measurement to calculate another quantity:

$$U = mgh \quad , \quad \begin{aligned} m &= 0.519\,\text{kg} \\ g &= 9.8\,\text{m/s}^2 \\ h &= 1.0 \pm 0.1\,\text{cm} \end{aligned}$$

Central value of U determined by plugging in:

$$\bar{U} = (0.519\,\text{kg})(9.8\,\text{m/s}^2)(1.0 \times 10^{-2}\,\text{m}) = 0.0508\,\text{J}$$

Error on U due to error on h?



$$\frac{\Delta U}{\Delta h} = \frac{dU}{dh} = mg$$

$$\boxed{\Delta U = mg\,\Delta h}$$

Now suppose we have errors on all three quantities.
Then we use partial derivatives and add the errors in quadrature

$$\Delta u = \sqrt{\left(\frac{\partial u}{\partial h}\Delta h\right)^2 + \left(\frac{\partial u}{\partial m}\Delta m\right)^2 + \left(\frac{\partial u}{\partial g}\Delta g\right)^2}$$

In general, if $F = F(x, y, z, \ldots)$, then

$$\Delta F = \sqrt{\left(\frac{\partial F}{\partial x}\Delta x\right)^2 + \left(\frac{\partial F}{\partial y}\Delta y\right)^2 + \left(\frac{\partial F}{\partial z}\Delta z\right)^2}$$

ΔF = absolute error, has same units as F

$\frac{\Delta F}{F}$ = fractional error, unitless.

Two simple cases:

If $F = a + b + c$, then $\Delta F = \sqrt{\Delta a^2 + \Delta b^2 + \Delta c^2}$

If $F = abc$, then $\left(\frac{\Delta F}{F}\right) = \sqrt{\left(\frac{\Delta a}{a}\right)^2 + \left(\frac{\Delta b}{b}\right)^2 + \left(\frac{\Delta c}{c}\right)^2}$

Waves

Typical Harmonic Wave: $\psi(x,t) = A\cos(kx - \omega t)$

A = amplitude

k = wave number $= \frac{2\pi}{\lambda}$, λ = wavelength

x = position

ω = angular frequency $= \frac{2\pi}{T}$, T = period

$= 2\pi\nu$, $\nu = \frac{1}{T}$ = frequency

t = time.

~~Wave velocity~~

Wave velocity : phase is constant

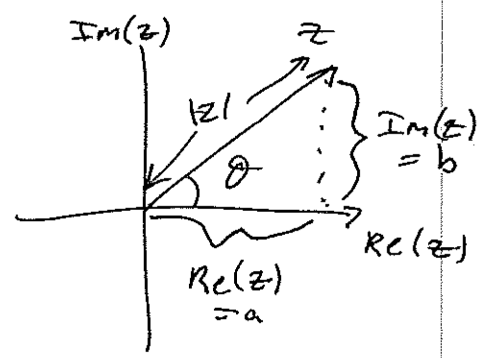
$$kx - \omega t = \text{constant}$$

$$k\frac{dx}{dt} - \omega = \phi$$

$$\frac{dx}{dt} = \frac{\omega}{k} = v = \frac{\lambda}{T} = \lambda\nu$$

Complex numbers

$z = a + ib, \quad i = \sqrt{-1}$



$Re(z) = a$

$Im(z) = b$

Magnitude of z: $|z| = \sqrt{a^2+b^2}$

$Re(z) = |z|\cos\theta = a$

$Im(z) = |z|\cos\theta = b$

$\theta = \tan^{-1}\left(\frac{b}{a}\right)$

$$z = |z|\underbrace{(\cos\theta + i\sin\theta)}_{e^{i\theta}\text{ by Euler's formula}} = |z|e^{i\theta}$$

Two ways to express a complex number:

$z = a + ib$, or $z = |z|e^{i\theta}$, $\theta = \tan^{-1}\left(\frac{b}{a}\right)$

Complex conjugate : $z^* = a - ib$ or $|z|e^{-i\theta}$.

$$zz^* = a^2+b^2 = |z|^2$$

Write the harmonic wave in complex form:

$$\psi(x,t) = A\cos(kx - \omega t) = \cancel{A}\, Re\left[Ae^{i(kx-\omega t)}\right]$$

$$\sim Ae^{i(kx-\omega t)}$$

remember to take the real part