Error Analysis

Two Types:

1. Statistical - errors which vary randomly from measurement to measurement.
   - Devices, instruments, apparatuses...
   - random errors
   - cumulative error.

2. Systematic - errors which bias measurement.
   - offset of the device from true value.
   - electronic noise if apparatus, inaccuracy of instrument.

Suppose we measure the period of a pendulum many times.

Statistical error - errors which vary randomly from measurement to measurement.

Systematic error - offsets from true value.

Sources of error:
- limitations of apparatuses
- human error
- measurement time
- electronic noise

Evaluate statistical errors by repeating the measurement.

Analyse carefully about the measurement.
A set of measurements \( \{x_i\} \)

Mean \( x = \bar{x} = \frac{\sum x_i}{N} \)

Median \( \tilde{x} \rightarrow \) a value such that one half of the \( x_i \) are above and one half below.

Standard Deviation = statistical error = Root-Mean-Square Dev.

\[
S = \sqrt{\frac{\sum (x_i - \mu)^2}{N}} \quad \mu = \text{true value of } x \text{, unknown to the experimenter}
\]

Variance \( = S^2 \)

\[
\sigma = \sqrt{\frac{\sum (x_i - \bar{x})^2}{N-1}} \quad \bar{x} = \text{mean value of } x_i \text{, can be calculated.}
\]

the usual way to evaluate statistical error.

If the errors are truly random, then the histogram will appear gaussian:

Full width at half maximum = 2.35 \( \sigma \) (FWHM)

68\% of measurements are within \( \pm 1\sigma \) from \( \bar{x} \).

95\% within \( \pm 2\sigma \)

99.7\% within \( \pm 3\sigma \)
Standard Deviation of the Mean:

What's the error on $\bar{x}$? Answer:

Typically we report a measurement and its statistical error as

$$\bar{x} \pm \frac{\sigma_{\bar{x}}}{\sqrt{N}} \pm \text{systematic error}$$

$\uparrow$ statistical error

Propagation of Errors

Suppose we use a measurement to calculate another quantity:

$$U = m g h$$  \quad m = 0.519 \text{ kg}$$

$$g = 9.8 \text{ m/s}^2$$

$$h = 1.0 \pm 0.1 \text{ cm}$$

Central value of $U$ determined by plugging in:

$$\bar{U} = \left(0.519 \text{ kg}\right)\left(9.8 \text{ m/s}^2\right)\left(1.0 \times 10^{-2} \text{ m}\right) = 0.0508 \text{ J}$$

Error on $U$ due to error in $h$?

$$\frac{\Delta U}{\Delta h} = \frac{dU}{dh} = mg$$

$$\Delta U = mg \Delta h$$
Now suppose we have errors on all three quantities. Then we use partial derivatives and add the errors in quadrature:

\[ \Delta U = \sqrt{\left( \frac{\partial U}{\partial h} \Delta h \right)^2 + \left( \frac{\partial U}{\partial m} \Delta m \right)^2 + \left( \frac{\partial U}{\partial g} \Delta g \right)^2} \]

In general, if \( F = F(x, y, z, \ldots) \), then

\[ \Delta F = \sqrt{\left( \frac{\partial F}{\partial x} \Delta x \right)^2 + \left( \frac{\partial F}{\partial y} \Delta y \right)^2 + \left( \frac{\partial F}{\partial z} \Delta z \right)^2} \]

\( \Delta F = \) absolute error, has same units as \( F \)

\( \frac{\Delta F}{F} = \) fractional error, unitless.

Two simple cases:

- If \( F = a + b + c \), then \( \Delta F = \sqrt{\Delta a^2 + \Delta b^2 + \Delta c^2} \)
- If \( F = abc \), then \( \left( \frac{\Delta F}{F} \right) = \sqrt{\left( \frac{\Delta a}{a} \right)^2 + \left( \frac{\Delta b}{b} \right)^2 + \left( \frac{\Delta c}{c} \right)^2} \)

Waves

Typical Harmonic Wave: \( \psi(x, t) = A \cos(kx - \omega t) \)

- \( A = \) amplitude
- \( k = \) wave number = \( \frac{2\pi}{\lambda} \), \( \lambda = \) wavelength
- \( x = \) position
- \( \omega = \) angular frequency = \( \frac{2\pi}{T} \), \( T = \) period
- \( \nu = \) wave velocity = \( \frac{v}{T} = \) frequency
- \( t = \) time.
Wave velocity: phase is constant
\[ kx - \omega t = \text{constant} \]
\[ k \frac{dx}{dt} - \omega = 0 \]
\[ \frac{dx}{dt} = \frac{\omega}{k} = v = \frac{\lambda}{T} = \lambda v \]

Complex numbers:
\[ z = a + ib, \quad i = \sqrt{-1} \]
\[ \text{Re}(z) = a \]
\[ \text{Im}(z) = b \]
Magnitude of \( z \): \[ |z| = \sqrt{a^2 + b^2} \]
\[ \text{Re}(z) = |z| \cos \theta = a \]
\[ \text{Im}(z) = |z| \sin \theta = b \]
\[ \theta = \tan^{-1} \left( \frac{b}{a} \right) \]
\[ z = |z| \left( \cos \theta + i \sin \theta \right) = |z| e^{i\theta} \]
\[ e^{i\theta} \text{ by Euler's formula} \]

Two ways to express a complex number:
\[ z = a + ib, \quad \text{or} \quad z = |z| e^{i\theta}, \quad \theta = \tan^{-1} \left( \frac{b}{a} \right) \]
Complex conjugate: \( \bar{z} = a - ib \) or \( |z| e^{-i\theta} \).
\[ z \bar{z} = a^2 + b^2 = |z|^2 \]

Write the harmonic wave in complex form:
\[ \Psi(x,t) = A \cos (kx - \omega t) = A \Re \left[ A e^{i(kx - \omega t)} \right] \]
\[ \sim A e^{i(kx - \omega t)} \]
\[ \Updownarrow \quad \text{remember to take the real part} \]