ERROR ANALYSIS FOR PHYS375

by
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1 Definition

Error: In a scientific measurement, an error means the inevitable uncertainty in the measured results. As such, errors are not mistakes. You can not avoid them by being careful. The best you can hope to do is to ensure that errors are as small as reasonably possible, and to have some reliable estimate of how large they are.

Discrepancy: If two measurements of the same quantity disagree, then we say that there is a discrepancy.

Note: One of the measurements could have been the so-called accepted value, value based on previous measurements as the "true" value, or a theoretically predicted value.

2 Two Types of Errors

Random or Statistical Errors: Experimental uncertainties that can be revealed by repeating the measurements are called random or statistical errors.

Systematic Errors: Experimental uncertainties that cannot be revealed by repeating the measurements are called systematic errors.

As an example, let's measure the well-defined width of a table top with a ruler. Uncertainty caused by needing to interpolate between scale markings is a random error. This is because when interpolating, one is probably just as likely to overestimate as to underestimate. On the other hand, uncertainty caused by the distortion of the ruler is a systematic error. This is because if the ruler has stretched, we always underestimate; if the ruler has shrunk, we always overestimate.

The treatment of random errors is quite different from that of systematic errors. The statistical methods to be discussed later give a reliable estimate of the random uncertainties, and, as we shall see, provide a well-defined procedure for reducing them. On the other hand, experienced scientists have to learn to anticipate the possible sources of systematic error, and to make sure that all systematic errors are much less than the required precision. Doing so will involve, for example, checking the instruments against accepted standards (or calibrated ones), and correcting them, or buying better instruments if necessary.

3 The Mean and Standard Deviation

Suppose we need to measure a quantity x and have identified all sources of systematic errors and reduced them to a negligible level. Since all remaining sources of uncertainty are random, we should be able to detect them by repeating the measurement several times. Suppose we make N (where $N \to \infty$) measurements of the quantity x (all using the same equipment and procedures), and find the N values:

$$x_1, x_2, \cdots, x_N$$

The best estimate for x is the average of $x_1, x_2, \cdots, x_N$, *i.e.*,

$$x_{best} = \bar{x} = \frac{x_1 + x_2 + \cdots + x_N}{N} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

We will also define the following quantities:

$$\begin{aligned} \sigma_x^2 &= \text{population variance} = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})^2 \\ \sigma_x &= \text{population standard deviation or rms (root-mean-square) deviation} \\ &= \lim_{N\to\infty} \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})^2} \end{aligned}$$

For finite N, it is more appropriate to define:

$$\begin{aligned} \sigma_x^2 \simeq s^2 &= \text{sample variance} = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2 \\ \sigma_x \simeq s &= \text{sample standard deviation} = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2} \end{aligned}$$

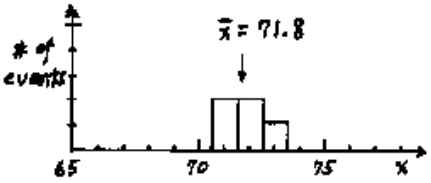
The factor $(N-1)$ is used in the sample variance and standard deviation instead of N. This is because we have to use data to find the mean $\bar{x}$. In a certain sense, this left only $(N-1)$ independent measured values. For large N, it does not make any difference either N or $(N-1)$ is used. For now, we will use σ_x to mean the sample standard deviation.

As an example, let's assume we have x_i : 71, 72, 72, 73, 71.

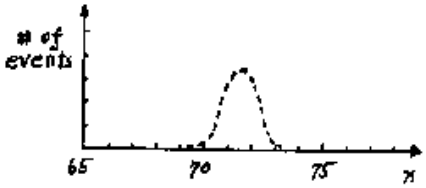
$$\begin{aligned} \bar{x} &= \frac{71+72+72+73+71}{5} = 71.8 \\ \sigma_x^2 &= \frac{(71.8-71)^2 + (71.8-72)^2 + \cdots}{5} = \frac{2.80}{5} = 0.56 \\ \sigma_x &\simeq 0.7 \\ s^2 &\simeq 0.7 \quad \Rightarrow \text{ write it as } \sigma_x^2 \\ s &\simeq 0.8 \quad \Rightarrow \text{ write it as } \sigma_x \end{aligned}$$

4 Meaning of the Standard Deviation - the Uncertainty in a Single Measurement

If we were to plot the above result as a histogram, we would have:

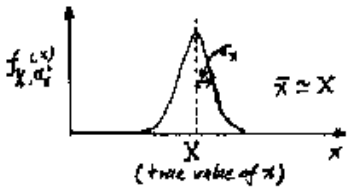


Instead of just making 5 measurements, if we were to make many measurements of x, we would get a limiting distribution as follows:



And the distribution could be represented by the so-called normal or Gaussian distribution:

$$f_{X,\sigma_x}(x) = \frac{1}{\sigma_x\sqrt{2\pi}} e^{-(x-X)^2/(2\sigma_x^2)}$$



Note that

$$\int_{-\infty}^{\infty} f_{X,\sigma_x}(x)dx = 1$$

Also

$$\int_{X-\sigma_x}^{X+\sigma_x} f_{X,\sigma_x}(x)dx \simeq 0.68 \Rightarrow 68\%$$

$$\int_{X-2\sigma_x}^{X+2\sigma_x} f_{X,\sigma_x}(x)dx \simeq 0.954 \Rightarrow 95.4\%$$

Let's suppose that we made N measurements of x and obtained the values $x_1, x_2, \cdots, x_N$. Let's compute $\bar{x}$ and σ_x. From the discussion above we can conclude that

> If our measurements are normally distributed and if we were to continue the measurement of x many more times (after making N measurements and using the same equipment), then about 68 % of our new measurements would lie within a distance σ_x on either side of $\bar{x}$; that is, 68 % of our new measurements would lie in the range $\bar{x} \pm \sigma_x$.

We can rephrase the above conclusion as follows:

Suppose, as before, that we obtain the values $x_1, x_2, \cdots, x_N$ and compute $\bar{x}$ and σ_x. If we then make one more measurement (using the same equipment), there is a 68 % probability that the new measurement will be within σ_x of $\bar{x}$. Now, if the original number of measurements N was large, then $\bar{x}$ should be a very reliable estimate for the actual value of x. Therefore, we can say that there is a 68 % probability that a single measurement (using the same equipment) will be within σ_x of the actual value.

5 The Standard Deviation of the Mean

If $x_1, x_2, \cdots, x_N$ are the results of N measurements of the same quantity x, then, as we have discussed earlier, our best estimate for the quantity x is their mean, $\bar{x}$. We have also discussed that the standard deviation σ_x characterizes the average uncertainty of the separate measurements $x_1, x_2, \cdots, x_N$. However, our answer $x_{best} = \bar{x}$ represents a judicious combination of all N measurements, and there is every reason to believe that $\bar{x}$ will be more reliable than any one of the measurements (x_i) considered separately. As we will show later, the uncertainty in the final answer $x_{best} = \bar{x}$ turns out to be the standard deviation σ_x divided by $\sqrt{N}$. This quantity is called the standard deviation of the mean, and is denoted by $\sigma_{\bar{x}}$:

$$\sigma_{\bar{x}} = \delta_x = \frac{\sigma_x}{\sqrt{N}}$$

We can now state our final answer for the value of x (based on the N measurements of x) as

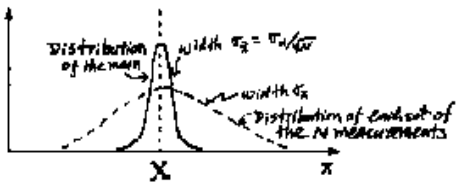
$$(\text{Value of } x) \ = \bar{x} \pm \frac{\sigma_x}{\sqrt{N}}$$

6 More on Standard Deviation and Standard Deviation of the Mean

Let's assume we have made many sets of N measurements of x with the same equipment:

$$\underbrace{x_1, x_2, \cdots, x_N}, \ \underbrace{x_1, x_2, \cdots, x_N}, \cdots$$

In other words, we have made many determinations of the average of N measurements. Each set of the N-measurements will be normally distributed about the true value X with width σ_x, shown as dashed curve below. The average of each set of the N-measurements will also be normally distributed about X, but with width $\sigma_{\bar{x}} = \sigma_x/\sqrt{N}$, shown as solid curve below.



7 Weighted Averages

It often happens that a physical quantity is measured several times, perhaps in several separate laboratories (or by different students), and the question arises how these measurements can be combined to give a single best estimate. Suppose, for example, that two students, A and B, measure a quantity x and obtain these results:

$$\begin{aligned} \text{Student A:} \quad & x = x_A \pm \sigma_A \\ \text{Student B:} \quad & x = x_B \pm \sigma_B \end{aligned}$$

Each of these results will probably itself be the results of several measurements, in which case, x_A will be the mean of all $A's$ measurements and σ_A the standard deviation of that mean (and similarly for x_B and σ_B). The question is how best to combine x_A and x_B as a single best estimate for x. The answer to this question is to use the principle of maximum likelihood as follows:

Let's assume that both measurements are "correct" (more discussion on this later) and they are governed by the Gaussian distribution. Let's further assume that the unknown true value of x is X. Then the probability of A obtaining the particular value of x_A is:

$$P_X(x_A) \propto \frac{1}{\sigma_A} e^{-(x_A - X)^2/(2\sigma_A^2)}$$

Similarly, for B:

$$P_X(x_B) \propto \frac{1}{\sigma_B} e^{-(x_B - X)^2/(2\sigma_B^2)}$$

The probability that A finds the value x_A and B the value x_B is just the product of the two probabilities:

$$\begin{aligned} P_X(x_A, x_B) &= P_X(x_A) P_X(x_B) \\ &\propto \frac{1}{\sigma_A \sigma_B} e^{-\left[\frac{(x_A - X)^2}{2\sigma_A^2} + \frac{(x_B - X)^2}{2\sigma_B^2}\right]} \\ &= \frac{1}{\sigma_A \sigma_B} e^{-\frac{\chi^2}{2}} \end{aligned}$$

where

$$\chi^2 = \frac{(x_A - X)^2}{\sigma_A^2} + \frac{(x_B - X)^2}{\sigma_B^2}$$

$P_X(x_A, x_B)$ would be maximum if the exponent is a minimum:

$$\begin{aligned} \frac{\partial \chi^2}{\partial X} &= 0 = \frac{2(x_A - X)(-1)}{\sigma_A^2} + \frac{2(x_B - X)(-1)}{\sigma_B^2} \\ \Rightarrow \quad & \frac{x_A - X}{\sigma_A^2} + \frac{x_B - X}{\sigma_B^2} = 0 \\ \Rightarrow \quad & X = x_{best} = \frac{\frac{x_A}{\sigma_A^2} + \frac{x_B}{\sigma_B^2}}{\frac{1}{\sigma_A^2} + \frac{1}{\sigma_B^2}} \end{aligned}$$

Defining the weights as:

$$\omega_A = \frac{1}{\sigma_A^2} \text{ and } \omega_B = \frac{1}{\sigma_B^2}$$

we obtain

$$x_{best} = \frac{\omega_A x_A + \omega_B x_B}{\omega_A + \omega_B}$$

This analysis can be generalized to combine several measurements of the same quantity. Suppose we have N separate measurements of a quantity x,

$$x_1 \pm \sigma_1,\ x_2 \pm \sigma_2,\ \cdots,\ x_N \pm \sigma_N$$

Then

$$x_{best} = \frac{\sum_{i=1}^{N} \omega_i x_i}{\sum_{i=1}^{N} \omega_i}$$

where

$$\omega_i = \frac{1}{\sigma_i^2}$$

It is obvious to note that the larger the error σ_i the smaller the contribution of x_i to the mean.

It can be shown that

$$\sigma_{x_{best}} = \left(\sum_{i=1}^{N} \omega_i \right)^{-1/2}$$

A special case: If all σ_i's are equal, *i.e.*,

$$\sigma_1 = \sigma_2 = \cdots = \sigma_N = \sigma$$

then

$$x_{best} = \frac{\frac{1}{\sigma^2} \sum_{i=1}^{N} x_i}{\frac{1}{\sigma^2} N} = \frac{1}{N} \sum_{i=1}^{N} x_i = \bar{x}$$

and

$$\sigma_{x_{best}} = \left(\frac{1}{\sigma^2} N \right)^{-1/2} = \frac{\sigma}{\sqrt{N}}$$

This implies that if a quantity is measured N times, the error will be improved over the error of a single measurement by a factor of $\frac{1}{\sqrt{N}}$. This is what we learn when we discussed the error of the mean earlier.

8 Consistency of the Data

After calculated the weighted average and the error, we can then calculate the χ^2:

$$\chi^2 = \sum_{i=1}^{N} \omega_i (x_i - x_{best})^2 = \sum_{i=1}^{N} \frac{(x_i - x_{best})^2}{\sigma_i^2}$$

and compare it with $N - 1$, which is the expectation value of χ^2 if the measurements are from a Gaussian distribution. We have the following three cases:

- If $\chi^2/(N-1)$ is less than or equal to 1 and there are no known problems with the data, we say that the data are consistent and should accept the results.
- If $\chi^2/(N-1)$ is greater than 1, but not greatly so, we can still accept the weighted average, but then we need to increase the error $\sigma_{x_{best}}$ by a scale factor defined as
$$S = \left[\chi^2/(N-1)\right]^{1/2}.$$
- If $\chi^2/(N-1)$ is very large, we say that the data are inconsistent and should suspect that something has gone wrong in at least one of the measurements. In this case, we should examine all the measurements carefully to see whether some (or all) of the measurements might be subject to unnoticed systematic errors (this could result in larger total errors than quoted). We may choose not to use the weighted average at all. Alternatively, we may quote the weighted average, but then make an educated guess of the error. For example, one could use the standard deviation
$$\sigma_{x_{best}} = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - x_{best})^2}$$
as the error for each measurement instead of the original individual σ_i, and use $\sigma_{x_{best}}/\sqrt{N}$ as the error in the weighted mean, instead of the weighted error
$$\sigma_{x_{best}} = \left(\sum_{i=1}^{N}\omega_i\right)^{-1/2}$$

9 Propagation of Errors

Suppose that, in order to find a value for the function $q(x,y)$, we measure the two quantities x and y several times, obtaining N pairs of data, (x_1,y_1), (x_2,y_2), $\cdots$, (x_n,y_N). From the N measurements x_1, x_2, $\cdots$, x_N, we can compute the mean $\bar{x}$ and standard deviation σ_x in the usual way; similarly, from y_1, y_2, $\cdots$, y_N, we can compute $\bar{y}$ and σ_y. Next, using the N pairs of measurements we can compute N values of the quantity of interest:
$$q_i = q(x_i,y_i), \qquad (i = 1,2,\cdots,N)$$
Given q_1, q_2, $\cdots$, q_N, we can now calculate their mean $\bar{q}$, which we assume giving our best estimate for q_i, and their standard deviation σ_q, which is our measure of the random uncertainty in the value of q_i.

We will assume, as usual, that all our uncertainties are small, and hence that all the numbers x_1, x_2, $\cdots$, x_N are close to $\bar{x}$ and that all the y_1, y_2, $\cdots$, y_N are close to $\bar{y}$. We can then use Taylor series expansion to make the approximation:
$$\begin{aligned} q_i &= q(x_i,y_i) \\ &\simeq q(\bar{x},\bar{y}) + \left.\frac{\partial q}{\partial x}\right|_{\bar{x},\bar{y}}(x_i-\bar{x}) + \left.\frac{\partial q}{\partial y}\right|_{\bar{x},\bar{y}}(y_i-\bar{y}) \end{aligned}$$

And we have

$$\begin{aligned}
\bar{q} &= \frac{1}{N}\sum_{i=1}^{N} q_i \\
&= \frac{1}{N}\sum_{i=1}^{N}\left[q(\bar{x},\bar{y}) + \left.\frac{\partial q}{\partial x}\right|_{\bar{x},\bar{y}}(x_i-\bar{x}) + \left.\frac{\partial q}{\partial y}\right|_{\bar{x},\bar{y}}(y_i-\bar{y})\right] \\
&= \frac{1}{N}\sum_{i=1}^{N} q(\bar{x},\bar{y}) + \frac{1}{N}\sum_{i=1}^{N}\left.\frac{\partial q}{\partial x}\right|_{\bar{x},\bar{y}}(x_i-\bar{x}) + \frac{1}{N}\sum_{i=1}^{N}\left.\frac{\partial q}{\partial y}\right|_{\bar{x},\bar{y}}(y_i-\bar{y}) \\
&= q(\bar{x},\bar{y})\frac{1}{N}\underbrace{\sum_{i=1}^{N} 1}_{N} + \left.\frac{\partial q}{\partial x}\right|_{\bar{x},\bar{y}}\frac{1}{N}\underbrace{\sum_{i=1}^{N}(x_i-\bar{x})}_{0} + \left.\frac{\partial q}{\partial y}\right|_{\bar{x},\bar{y}}\frac{1}{N}\underbrace{\sum_{i=1}^{N}(y_i-\bar{y})}_{0} \\
&= q(\bar{x},\bar{y})
\end{aligned}$$

This means, to find the mean $\bar{q}$, we have only to calculate the function $q(x,y)$ at the point $x=\bar{x}$ and $y=\bar{y}$.

The standard deviation of the N values q_1, q_2, $\cdots$, q_N is given by

$$\begin{aligned}
\sigma_q^2 &= \frac{1}{N}\sum_{i=1}^{N}(q_i-\bar{q})^2 \\
&= \frac{1}{N}\sum_{i=1}^{N}\left[\left.\frac{\partial q}{\partial x}\right|_{\bar{x},\bar{y}}(x_i-\bar{x}) + \left.\frac{\partial q}{\partial y}\right|_{\bar{x},\bar{y}}(y_i-\bar{y})\right]^2 \\
&= \left(\frac{\partial q}{\partial x}\right)^2_{\bar{x},\bar{y}}\underbrace{\frac{1}{N}\sum_{i=1}^{N}(x_i-\bar{x})^2}_{\sigma_x^2} + \left(\frac{\partial q}{\partial y}\right)^2_{\bar{x},\bar{y}}\underbrace{\frac{1}{N}\sum_{i=1}^{N}(y_i-\bar{y})^2}_{\sigma_y^2} \\
&\quad +2\left.\frac{\partial q}{\partial x}\right|_{\bar{x},\bar{y}}\left.\frac{\partial q}{\partial x}\right|_{\bar{x},\bar{y}}\frac{1}{N}\sum_{i=1}^{N}(x_i-\bar{x})(y_i-\bar{y})
\end{aligned}$$

The first two terms are just σ_x^2 and σ_y^2. Let's define the so-called the covariance of x and y as follows:

$$\sigma_{xy} = \frac{1}{N}\sum_{i=1}^{N}(x_i-\bar{x})(y_i-\bar{y})$$

then we have

$$\sigma_q^2 = \left(\frac{\partial q}{\partial x}\right)^2\sigma_x^2 + \left(\frac{\partial q}{\partial y}\right)^2\sigma_y^2 + 2\left(\frac{\partial q}{\partial x}\right)\left(\frac{\partial q}{\partial y}\right)\sigma_{xy}$$

(Note that we have dropped the subscripts $\bar{x}$ and $\bar{y}$.)

If x and y are independent, then $\sigma_{xy}=0$. This is because for a given value of y_i, the quantity $(x_i-\bar{x})$ is just as likely to be negative as it is to be positive. (This is true for any given value of y_i.) Thus, after many measurements, the positive and negative terms in σ_{xy} should nearly balance. We then have

$$\sigma_q^2 = \left(\frac{\partial q}{\partial x}\right)^2\sigma_x^2 + \left(\frac{\partial q}{\partial y}\right)^2\sigma_y^2$$

or

$$\sigma_q = \sqrt{\left(\frac{\partial q}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial q}{\partial y}\right)^2 \sigma_y^2}$$

If there are more than two variables, then we have

$$\sigma_q = \sqrt{\left(\frac{\partial q}{\partial x}\right)^2 \sigma_x^2 + \cdots + \left(\frac{\partial q}{\partial z}\right)^2 \sigma_z^2}$$

10 Specific Formulas

10.1 Sums and Differences

Let $q = ax \pm by$, then

$$\begin{aligned} \frac{\partial q}{\partial x} &= a, \qquad \frac{\partial q}{\partial y} = \pm b \\ \sigma_q &= \sqrt{(a)^2\sigma_x^2 + (\pm b)^2\sigma_y^2} \\ &= \sqrt{a^2\sigma_x^2 + b^2\sigma_y^2} \qquad \leftarrow \text{ added in quadrature} \end{aligned}$$

10.2 Products and Quotients

- For products: Let $q = axy$, then

$$\begin{aligned} \frac{\partial q}{\partial x} &= ay \\ \frac{\partial q}{\partial y} &= ax \\ \sigma_q &= \sqrt{(ay)^2\sigma_x^2 + (ax)^2\sigma_y^2} \\ &= \sqrt{\frac{q^2}{a^2x^2y^2}(a^2y^2\sigma_x^2 + a^2x^2\sigma_y^2)} \\ &= q\sqrt{\left(\frac{\sigma_x}{x}\right)^2 + \left(\frac{\sigma_y}{y}\right)^2} \\ \Rightarrow \frac{\sigma_q}{q} &= \sqrt{\left(\frac{\sigma_x}{x}\right)^2 + \left(\frac{\sigma_y}{y}\right)^2} \end{aligned}$$

  $\Rightarrow$ Added in quadrature for the fractional errors.

- For quotients: Let $q = a\frac{x}{y}$, then

$$\frac{\partial q}{\partial x} = \frac{a}{y}$$

$$\begin{aligned}
\frac{\partial q}{\partial y} &= -\frac{ax}{y^2} \\
\sigma_q &= \sqrt{\left(\frac{a}{y}\right)^2 \sigma_x^2 + \left(-\frac{ax}{y^2}\right)^2 \sigma_y^2} \\
&= \sqrt{\frac{q^2y^2}{a^2x^2}\left(\frac{a^2\sigma_x^2}{y^2} + \frac{a^2x^2\sigma_y^2}{y^4}\right)} \\
&= q\sqrt{\left(\frac{\sigma_x}{x}\right)^2 + \left(\frac{\sigma_y}{y}\right)^2} \\
\Rightarrow \frac{\sigma_q}{q} &= \sqrt{\left(\frac{\sigma_x}{x}\right)^2 + \left(\frac{\sigma_y}{y}\right)^2}
\end{aligned}$$

$\Rightarrow$ Added in quadrature for the fractional errors.

10.3 Powers

Let $q = ax^{\pm n}$, then

$$\begin{aligned}
\frac{\partial q}{\partial x} &= a(\pm n)x^{\pm n-1} = \pm\frac{nq}{x} \\
\Rightarrow \sigma_q &= \sqrt{\left(\pm\frac{nq}{x}\right)^2 \sigma_x^2} \\
&= \sqrt{\frac{n^2q^2}{x^2}\sigma_x^2} \\
&= q\sqrt{n^2\left(\frac{\sigma_x}{x}\right)^2} \\
\Rightarrow \frac{\sigma_q}{q} &= n\left(\frac{\sigma_x}{x}\right)
\end{aligned}$$

$\Rightarrow$ The fractional error in x is increased by a factor n.

10.4 Exponentials

Let $q = ae^{\pm bx}$, then

$$\begin{aligned}
\frac{\partial q}{\partial x} &= a(\pm b)e^{\pm bx} = \pm bq \\
\sigma_q &= \sqrt{(\pm bq)^2\sigma_x^2} = bq\sigma_x \\
\Rightarrow \frac{\sigma_q}{q} &= b\sigma_x
\end{aligned}$$

10.5 Logarithm

Let $q = a\ln(\pm bx)$, then

$$\begin{aligned}\frac{\partial q}{\partial x} &= a\left(\frac{\pm b}{\pm bx}\right) = \frac{a}{x} \\ \sigma_q &= \sqrt{\left(\frac{a}{x}\right)^2 \sigma_x^2} = \frac{a}{x}\sigma_x \\ &= a\left(\frac{\sigma_x}{x}\right)\end{aligned}$$

10.6 Three Examples

1. Measurement of g, the acceleration of gravity, using a simple pendulum. We have

$$\begin{aligned}T &= 2\pi\sqrt{\frac{l}{g}} \\ \Rightarrow\ g &= \frac{4\pi^2 l}{T^2}\end{aligned}$$

Using the formulae given above, we have

$$\frac{\sigma_g}{g} = \sqrt{\left(\frac{\sigma_l}{l}\right)^2 + \left(\frac{\sigma_{T^2}}{T^2}\right)^2}$$

But,

$$\begin{aligned}\frac{\sigma_{T^2}}{T^2} &= 2\,\frac{\sigma_T}{T} \\ \Rightarrow\ \frac{\sigma_g}{g} &= \sqrt{\left(\frac{\sigma_l}{l}\right)^2 + \left(2\,\frac{\sigma_T}{T}\right)^2}\end{aligned}$$

Suppose

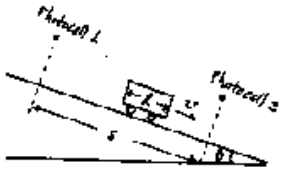
$$\begin{aligned}l &= 92.95 \pm 0.1\,cm \\ T &= 1.936 \pm 0.004\,sec \\ \Rightarrow\ g_{best} &= \frac{4\pi^2 \times (92.95)}{(1.936)^2} = 979\,cm/sec^2\end{aligned}$$

We have

$$\begin{aligned}\frac{\sigma_l}{l} &= \frac{0.1}{92.95} = 0.1\% \\ \frac{\sigma_T}{T} &= \frac{0.004}{1.936} = 0.2\% \\ \frac{\sigma_g}{g} &= \sqrt{(0.1\%)^2 + (2 \times 0.2\%)^2} = 0.4\% \\ \Rightarrow\ \sigma_g &= 0.004 \times 979 = 4\,cm/sec^2 \\ \Rightarrow\ g &= (979 \pm 4)\,cm/sec^2\end{aligned}$$

Note: There is no need to improve the accuracy in the measurement of l since final error is dominated by the error in T.

2. Acceleration of a cart down a slope. We have



$$\begin{aligned} v_2^2 &= v_1^2 + 2as \\ a &= \frac{v_2^2 - v_1^2}{2s} \end{aligned}$$

But

$$v_1 = \frac{l}{t_1}, \qquad v_2 = \frac{l}{t_2}$$

$$\Rightarrow a = \left(\frac{l^2}{2s}\right)\left(\frac{1}{t_2^2} - \frac{1}{t_1^2}\right)$$

We assume

$$\begin{aligned} l &= 5.0 \pm 0.05\,cm \quad (1\%) \\ s &= 100.0 \pm 0.2\,cm \quad (0.2\%) \\ t_1 &= 0.054 \pm 0.001\,sec \quad (2\%) \\ t_2 &= 0.031 \pm 0.001\,sec \quad (3\%) \end{aligned}$$

To calculate σ_a, let's do it in steps.

(a) First, let's find $\sigma_{\left(\frac{l^2}{2s}\right)}$:

$$\begin{aligned} \frac{\sigma_{\left(\frac{l^2}{2s}\right)}}{\left(\frac{l^2}{2s}\right)} &= \sqrt{\left(\frac{\sigma_{l^2}}{l^2}\right)^2 + \left(\frac{\sigma_s}{s}\right)^2} \\ &= \sqrt{\left(\frac{2\sigma_l}{l}\right)^2 + \left(\frac{\sigma_s}{s}\right)^2} \\ &= \sqrt{(2 \times 1\%)^2 + (0.2\%)^2} \\ &= 2\% \end{aligned}$$

Note that the uncertainty in s makes no appreciable contribution.

(b) Next, let's find $\sigma_{\left(\frac{1}{t_2^2}\right)}$ and $\sigma_{\left(\frac{1}{t_1^2}\right)}$:

$$\begin{aligned} \sigma_{\left(\frac{1}{t_1^2}\right)} &= 2\sigma_{\left(\frac{1}{t_1}\right)} = 2\sigma_{t_1} = 2 \times 2\% = 4\% \\ \sigma_{\left(\frac{1}{t_2^2}\right)} &= 2\sigma_{\left(\frac{1}{t_2}\right)} = 2\sigma_{t_2} = 2 \times 3\% = 6\% \end{aligned}$$

$$\Rightarrow \frac{1}{t_1^2} = (343 \pm \underbrace{14}_{4\%})\, sec^{-2}$$

$$\Rightarrow \frac{1}{t_2^2} = (1041 \pm \underbrace{62}_{6\%})\, sec^{-2}$$

(c) Next, let's find $\sigma_{\left(\frac{1}{t_2^2}-\frac{1}{t_1^2}\right)}$:

$$\begin{aligned} \sigma_{\left(\frac{1}{t_2^2}-\frac{1}{t_1^2}\right)} &= \sqrt{\sigma^2_{\left(\frac{1}{t_2^2}\right)} + \sigma^2_{\left(\frac{1}{t_1^2}\right)}} \\ &= \sqrt{(14)^2 + (62)^2} \\ &= 64\, sec^{-2} \\ \Rightarrow \frac{1}{t_2^2} - \frac{1}{t_1^2} &= (698 \pm 64)\, sec^{-2} \quad (9\%) \end{aligned}$$

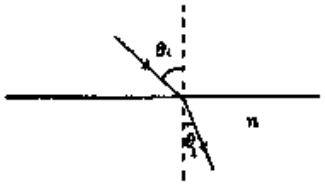
(d) Lastly,

$$\begin{aligned} \frac{\sigma_a}{a} &= \sqrt{\left(\frac{\sigma_{\left(\frac{l^2}{2s}\right)}}{\frac{l^2}{2s}}\right)^2 + \left(\frac{\sigma_{\left(\frac{1}{t_2^2}-\frac{1}{t_1^2}\right)}}{\frac{1}{t_2^2}-\frac{1}{t_1^2}}\right)^2} \\ &= \sqrt{(2\%)^2 + (9\%)^2} \\ &= 9\% \\ \Rightarrow a &= \left(\frac{5 \times 5}{2 \times 100}\right) \times 698 \pm 9\% \\ &= 87.3\, cm/sec^2 \pm 9\% \\ &= (87.3 \pm 8)\, cm/sec^2 \end{aligned}$$

3. Refractive index using Snell's law. We have

$$n = \frac{\sin\theta_1}{\sin\theta_2}$$

where θ_1 (the angle of incidence) and θ_2 (the angle of refraction) are measured.



$$\frac{\sigma_n}{n} = \sqrt{\left(\frac{\sigma_{\sin\theta_1}}{\sin\theta_1}\right)^2 + \left(\frac{\sigma_{\sin\theta_2}}{\sin\theta_2}\right)^2}$$

To find $\frac{\sigma_{\sin\ \alpha}}{\sin\ \alpha}$, let's use

$$\begin{aligned} \sigma_q &= \sqrt{\left(\frac{\partial q}{\partial x}\right)^2 \sigma_x^2} \quad \Leftarrow \text{use original formula} \\ \Rightarrow \frac{\partial \sin\ \alpha}{\partial \alpha} &= \cos\ \alpha \end{aligned}$$

And

$$\begin{aligned} \sigma_{\sin\theta_1} &= \cos\theta_1\sigma_{\theta_1} \\ \frac{\sigma_{\sin\theta_1}}{\sin\theta_1} &= \frac{\cos\theta_1}{\sin\theta_1}\sigma_{\theta_1} = \cot\theta_1\sigma_{\theta_1} \end{aligned}$$

Similarly, we have

$$\begin{aligned} \frac{\sigma_{\sin\theta_2}}{\sin\theta_2} &= \cot\theta_2\sigma_{\theta_2} \\ \Rightarrow \frac{\sigma_n}{n} &= \sqrt{\cot^2\theta_1\sigma_{\theta_1}^2 + \cot^2\theta_2\sigma_{\theta_2}^2} \end{aligned}$$

Suppose we now measure the angle θ_2 for a couple of values of θ_1, and get the results shown in the first two columns of the table below. Let's assume that all angles are measured to an accuracy of $\pm 1^o$, or 0.0175 radians. And the results are:

θ_1(deg)	θ_2(deg)	n	$\cot\theta_1\sigma_{\theta_1}$	$\cot\theta_2\sigma_{\theta_2}$	$\frac{\sigma_n}{n}$	σ_n
20 ± 1	13 ± 1	1.52	5%*	8%†	9%	0.14
40 ± 1	23.5 ± 1	1.61	2%	4%	5%	0.08

$$\begin{aligned} * \quad \cot(20) \times 0.0175 &= 0.05 = 5\% \\ \dagger \quad \cot(13) \times 0.0175 &= 0.08 = 8\% \end{aligned}$$

And

$$\begin{aligned} \bar{n} &= \frac{\frac{1.52}{(0.14)^2} + \frac{1.61}{(0.08)^2}}{\frac{1}{(0.14)^2} + \frac{1}{(0.08)^2}} = 1.59 \\ \sigma_{n_{best}} &= \left[\frac{1}{\frac{1}{(0.14)^2} + \frac{1}{(0.08)^2}}\right]^{1/2} = 0.07 \end{aligned}$$

Thus,

$$\text{(Value of n)} = 1.59 \pm 0.07$$

11 Least-Aquare Fit to a Straight Line

It often happens that we wish to determine one characteristic of an experiment y as a function of some other quantity x. That is, we wish to find the function f such that $y = f(x)$. Instead of making a number of measurements of the quantity y for one particular value of x (of course, one does this so that one can determine σ_y), we make a series of N measurements y_i, one for each of several values of the quantity $x = x_i$, where i is an index that runs from 1 to N to identify the measurements.

Probably the most important experiments of this type are those where the expected relation (function) is linear, and this is the case we consider here. In other words, our data consist of pairs of measurements (x_i, y_i). We wish to fit the data with an equation of the form:

$$y = a + bx$$

by determining the values of the coefficients a and b such that the discrepancy is minimized between the values of our measured y_i and the corresponding values $y_i = f(x_i)$.

The problem is to establish and to optimize the estimates of the coefficients. We will again use the principle of maximum likelihood for this problem.

If we knew the constants a and b, then, for any given value x_i (which we assume to have no uncertainty), we could compute the true value of the corresponding y_i:

$$(\text{True value for } y_i) \ = a + bx_i$$

Assuming that the measurement of y_i is governed by a normal (Gaussian) distribution centered on this true value, with a width σ_{y_i}. Therefore, the probability of obtaining the observed value y_i is

$$P_{a,b}(y_i) \propto \frac{1}{\sigma_{y_i}} e^{-(y_i - a - bx_i)^2/(2\sigma_{y_i}^2)}$$

where the subscripts a and b indicate that this probability depends on the unknown values of a and b. The probability of obtaining our complete set of measurements (x_1, y_1), (x_2, y_2), $\cdots$, (x_N, y_N) is the product:

$$\begin{aligned} P_{a,b}(y_1, y_2, \cdots, y_N) &\propto P_{a,b}(y_1) P_{a,b}(y_2) \cdots P_{a,b}(y_N) \\ &\propto \frac{1}{\sigma_{y_1}\sigma_{y_2}\cdots\sigma_{y_N}} e^{\left\{-\frac{1}{2}\sum_i \frac{(y_i - a - bx_i)^2}{\sigma_{y_i}^2}\right\}} \\ &= \frac{1}{\sigma_{y_1}\sigma_{y_2}\cdots\sigma_{y_N}} e^{-\frac{\chi^2}{2}} \end{aligned}$$

where

$$\chi^2 = \sum_{i=1}^{N} \frac{(y_i - a - bx_i)^2}{\sigma_{y_i}^2}$$

As before, the best estimate for the unknown constants a and b, based on the given measurements, are those values of a and b for which the probability $P_{a,b}(y_1, y_2, \cdots, y_N)$ is maximum or for which χ^2 (the sum of squares) is a minimum (that is why this method is known as the least-squares fitting). To find the values of a and b which yield the minimum value of χ^2, we differentiate χ^2 with respect to a and b and set the derivatives to zero:

$$\begin{aligned} \frac{\partial \chi^2}{\partial a} &= -2\sum_{i=1}^{N} \frac{(y_i - a - bx_i)}{\sigma_{y_i}^2} = 0 \\ \frac{\partial \chi^2}{\partial b} &= -2\sum_{i=1}^{N} \frac{(y_i - a - bx_i)x_i}{\sigma_{y_i}^2} = 0 \end{aligned}$$

These equations can be rearranged to yield a pair of simultaneous equations:

$$\begin{aligned}\sum_{i=1}^{N}\frac{y_i}{\sigma_{y_i}^2} &= a\sum_{i=1}^{N}\frac{1}{\sigma_{y_i}^2}+b\sum_{i=1}^{N}\frac{x_i}{\sigma_{y_i}^2}\\ \sum_{i=1}^{N}\frac{x_iy_i}{\sigma_{y_i}^2} &= a\sum_{i=1}^{N}\frac{x_i}{\sigma_{y_i}^2}+b\sum_{i=1}^{N}\frac{x_i^2}{\sigma_{y_i}^2}\end{aligned}$$

The solutions are:

$$\begin{aligned}a &= \frac{1}{\Delta}\begin{vmatrix}\sum_{i=1}^{N}\frac{y_i}{\sigma_{y_i}^2} & \sum_{i=1}^{N}\frac{x_i}{\sigma_{y_i}^2}\\ \sum_{i=1}^{N}\frac{x_iy_i}{\sigma_{y_i}^2} & \sum_{i=1}^{N}\frac{x_i^2}{\sigma_{y_i}^2}\end{vmatrix}\\ &= \frac{1}{\Delta}\left(\sum_{i=1}^{N}\frac{x_i^2}{\sigma_{y_i}^2}\sum_{i=1}^{N}\frac{y_i}{\sigma_{y_i}^2}-\sum_{i=1}^{N}\frac{x_i}{\sigma_{y_i}^2}\sum_{i=1}^{N}\frac{x_iy_i}{\sigma_{y_i}^2}\right)\\ b &= \frac{1}{\Delta}\begin{vmatrix}\sum_{i=1}^{N}\frac{1}{\sigma_{y_i}^2} & \sum_{i=1}^{N}\frac{y_i}{\sigma_{y_i}^2}\\ \sum_{i=1}^{N}\frac{x_i}{\sigma_{y_i}^2} & \sum_{i=1}^{N}\frac{x_iy_i}{\sigma_{y_i}^2}\end{vmatrix}\\ &= \frac{1}{\Delta}\left(\sum_{i=1}^{N}\frac{1}{\sigma_{y_i}^2}\sum_{i=1}^{N}\frac{x_iy_i}{\sigma_{y_i}^2}-\sum_{i=1}^{N}\frac{x_i}{\sigma_{y_i}^2}\sum_{i=1}^{N}\frac{y_i}{\sigma_{y_i}^2}\right)\\ \Delta &= \begin{vmatrix}\sum_{i=1}^{N}\frac{1}{\sigma_{y_i}^2} & \sum_{i=1}^{N}\frac{x_i}{\sigma_{y_i}^2}\\ \sum_{i=1}^{N}\frac{x_i}{\sigma_{y_i}^2} & \sum_{i=1}^{N}\frac{x_i^2}{\sigma_{y_i}^2}\end{vmatrix}\\ &= \sum_{i=1}^{N}\frac{1}{\sigma_{y_i}^2}\sum_{i=1}^{N}\frac{x_i^2}{\sigma_{y_i}^2}-\left(\sum_{i=1}^{N}\frac{x_i}{\sigma_{y_i}^2}\right)^2\end{aligned}$$

We state without proof that the errors in a and b are given as:

$$\begin{aligned}\sigma_a^2 &\cong \frac{1}{\Delta}\sum_{i=1}^{N}\frac{x_i^2}{\sigma_{y_i}^2}\\ \sigma_b^2 &\cong \frac{1}{\Delta}\sum_{i=1}^{N}\frac{1}{\sigma_{y_i}^2}\end{aligned}$$

Special case: Let's sssume that all the errors are equal, we have

$$\sigma_{y_1}=\sigma_{y_2}=\cdots=\sigma_{y_N}=\sigma_y$$

then

$$\begin{aligned}a &= \frac{1}{\Delta}\left(\sum_{i=1}^{N}x_i^2\sum_{i=1}^{N}y_i-\sum_{i=1}^{N}x_i\sum_{i=1}^{N}x_iy_i\right)\\ b &= \frac{1}{\Delta}\left(N\sum_{i=1}^{N}x_iy_i-\sum_{i=1}^{N}x_i\sum_{i=1}^{N}y_i\right)\end{aligned}$$

where

$$\Delta = N\sum_{i=1}^{N}x_i^2-\left(\sum_{i=1}^{N}x_i\right)^2$$

and

$$\sigma_a^2 = \frac{1}{\Delta}\left(\sigma_y^2 \sum_{i=1}^{N} x_i^2\right)$$
$$\sigma_b^2 = \frac{1}{\Delta}\left(N\sigma_y^2\right)$$

12 Uncertainty in the Measurement of y

If each of the y_i for a given x_i is measured many times, one can certainly get an idea of σ_{y_i} by examining the spread in their values, *i.e.*, the sample standard deviation. These values of σ_{y_i} can then be used in the least-squares fitting. On the other hand, if any pair of (x_i, y_i) is measured only once, one can not calculate σ_{y_i} by examining the spread in the measurements. This is because the numbers $y_1, y_2, \cdots, y_N$ are <u>not</u> N measurements of the same quantity (same x_i). Although one can not use these data to calculate the individual σ_{y_i} one can still make an estimate of the common uncertainty σ_y in the numbers $y_1, y_2, \cdots, y_N$ by analyzing the data themselves.

Let's assume that the measurement of each y_i is normally distributed about its true value $a + bx_i$ with width parameter σ_y. Thus, the deviations $y_i - a - bx_i$ are normally distributed, all with the same central value of 0 and with the same width σ_y. This immediately suggests that a good estimate for σ_y would be given by the sum of squares of deviations of the data points from the calculated mean divided by $(N-2)$:

$$\sigma_y^2 = \frac{1}{N-2}\sum_{i=1}^{N}(y_i - a - bx_i)^2$$

The presence of the factor $(N-2)$ is reminiscent of the $(N-1)$ factor that appeared in our estimate of the standard deviation of N measurements of one quantity x. There we made N measurements $x_1, x_2, \cdots, x_N$ of one quantity x. Before we could calculate σ_x, we had to use our data to find the mean $\bar{x}$. In a certain sense, this left only $(N-1)$ independent measured values; so we say that, having computed $\bar{x}$, we have only $(N-1)$ degrees of freedom left. Here we made N measurements, but before calculating σ_y we had to compute the two quantities a and b. Having done this, we had only $(N-2)$ degrees of freedom left.