# Superposition of waves

$$\nabla^2 \vec{E}_1 = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E}_1$$

If $\vec{E}_2$ is also a solution, $\nabla^2 \vec{E}_2 = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E}_2$

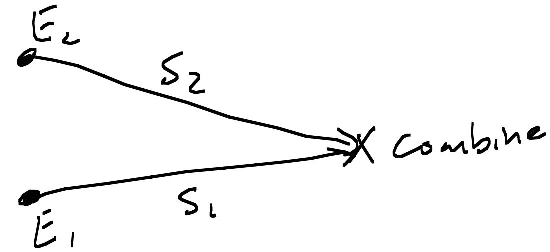
Then $\vec{E} = \vec{E}_1 + \vec{E}_2$ is also a solution:

$$\nabla^2 \left(\vec{E}_1 + \vec{E}_2\right) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(\vec{E}_1 + \vec{E}_2\right)$$

$$\underbrace{\nabla^2 \vec{E}_1} + \underbrace{\nabla^2 \vec{E}_2} = \underbrace{\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E}_1} + \underbrace{\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E}_2} \quad \checkmark$$

$=$ $=$

## Superposition of 2 waves (same K and polarization)




$$\vec{E}_1 = E_{01}\cos(\alpha_1 - \omega t) \qquad \vec{E}_2 = E_{02}\cos(\alpha_2 - \omega t)$$

$$\alpha_1 = Ks_1 + \phi_1 \qquad \alpha_2 = Ks_2 + \phi_2$$

$$E_1 + E_2 = E_{01}\cos(\alpha_1 - \omega t) + E_{02}\cos(\alpha_2 - \omega t)$$

$$= E_{01}\cos(\alpha_1 - \omega t) + E_{01}\cos(\alpha_2 - \omega t) + (E_{02} - E_{01})\cos(\alpha_2 - \omega t)$$

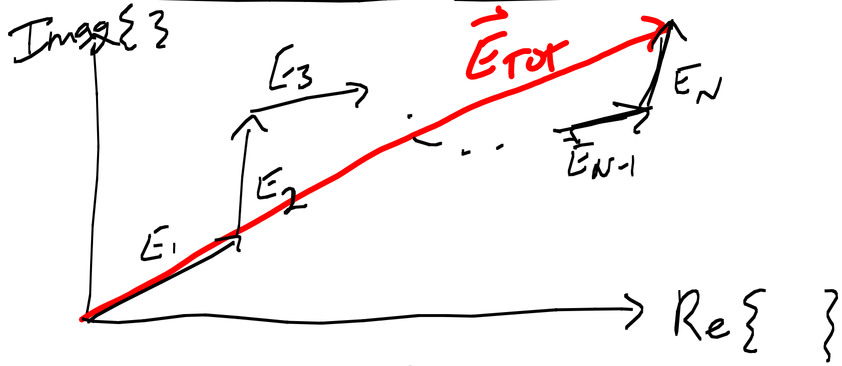
$$\boxed{\cos u + \cos v = 2\cos\frac{u+v}{2}\cos\frac{u-v}{2}}$$

$$= 2E_{01}\cos\left(\frac{\alpha_1 + \alpha_2 - 2\omega t}{2}\right)\cos\left(\frac{\alpha_1 - \alpha_2}{2}\right) + (E_{02} - E_{01})\cos(\alpha_2 - \omega t)$$

If $\alpha_1 - \alpha_2 = 0, 2\pi, 4\pi, \ldots\ 2m\pi \qquad m = 0, \pm 1, \pm 2 \ldots$ "constructive"

$= \pi, 3\pi, \ldots\ (2m+1)\pi$ "destructive"

## Superposition of N waves: phasors




$$\vec{E} \sim E e^{i(ks-\omega t)}$$

$$\vec{E}_{tot} = \left[\sum_i^N E_{0i} \cos\alpha_i\right] + i\left[\sum_i^N E_{0i} \sin\alpha_i\right]$$

$$(a+b)^2 = a^2+b^2 + ab + ba$$

$$I = |\vec{E}_{tot}|^2 = \left[\sum_i^N E_{0i} \cos\alpha_i\right]^2 + \left[\sum_i^N E_{0i} \sin\alpha_i\right]^2$$

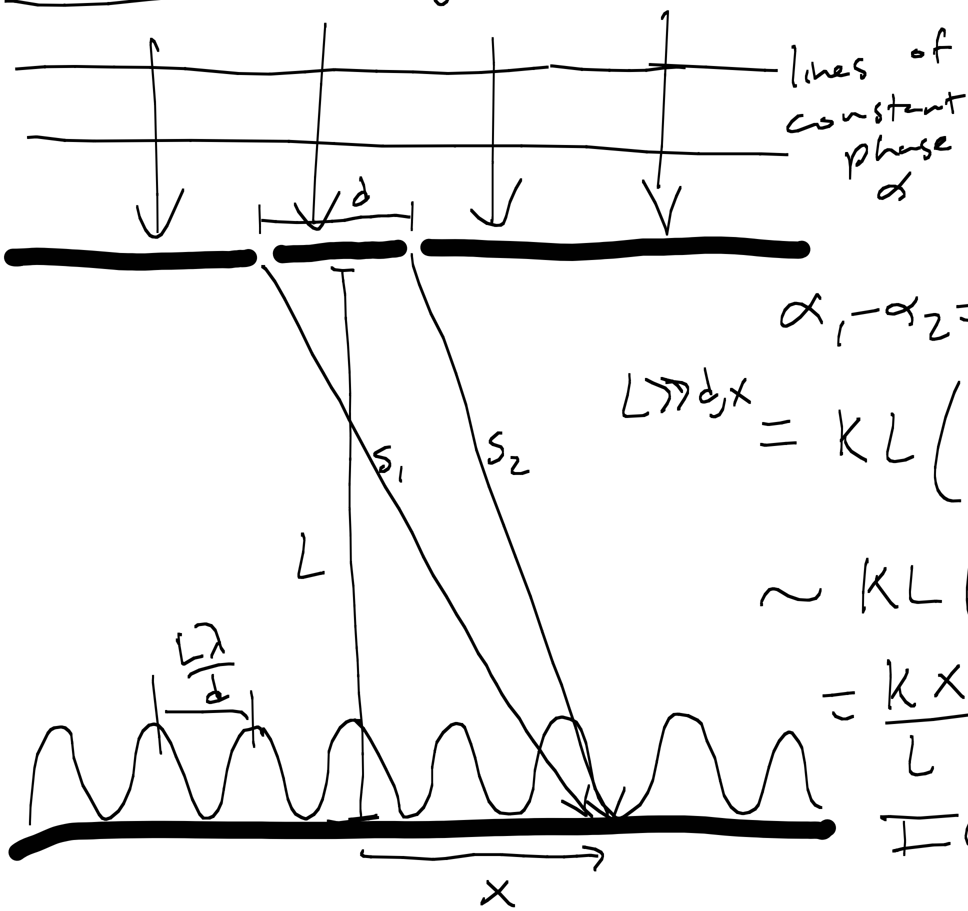
$$= \sum_{i=1}^N E_{0i}^2 \cos^2\alpha_i + \sum_{i=1}^N E_{0i}^2 \sin^2\alpha_i + 2\sum_{j>i}^N \sum_{i=1}^N E_{0i} E_{0j} \cos\alpha_i \cos\alpha_j + 2\sum_{j>i}^N \sum_{i=1}^N E_{0i} E_{0j} \sin\alpha_i \sin\alpha_j$$

$$\boxed{\cos(A-B) = \cos A \cos B + \sin A \sin B \quad \checkmark \qquad \sin^2\alpha + \cos^2\alpha = 1}$$

$$= \sum_{i=1}^N E_{0i}^2 + 2\sum_{j>i}^N \sum_{i=1}^N E_{0i} E_{0j} \cos(\alpha_j - \alpha_i)$$

Second term $\to 0$ when sources are incoherent!

# Example: Young's two-slit experiment




$$I(x) = \frac{I_0}{2} + \frac{I_0}{2} + 2\sqrt{\frac{I_0}{2}}\sqrt{\frac{I_0}{2}}\cos(\alpha_1-\alpha_2)$$

$$= I_0 + I_0\cos(\alpha_1-\alpha_2)$$

$$\alpha_1-\alpha_2 = k(s_1-s_2) = k\left(\sqrt{L^2+(x+\tfrac{d}{2})^2} - \sqrt{L^2+(x-\tfrac{d}{2})^2}\right)$$

$L \gg d, x$

$$= kL\left(\sqrt{1+\frac{(x+\frac{d}{2})^2}{L^2}} - \sqrt{1+\frac{(x-\frac{d}{2})^2}{L^2}}\right)$$

$$\sim kL\left(1+\frac{x^2+xd+\frac{d^2}{4}}{2L^2} - \left(1+\frac{x^2-xd+\frac{d^2}{4}}{2L^2}\right)\right)$$

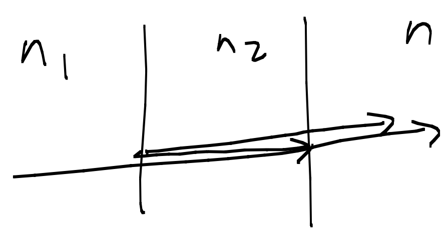
$$= \frac{kxd}{L}$$

$$I(x) = I_0\left(1+\cos\frac{kxd}{L}\right)$$

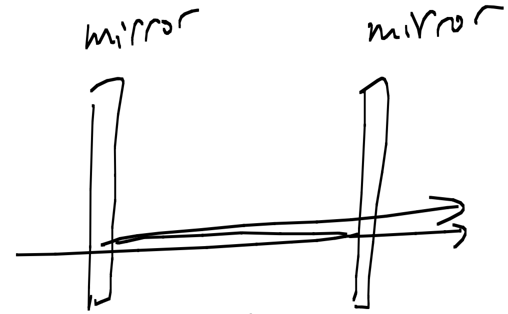
Bright spot when $\frac{kxd}{L} = 0, 2\pi, \ldots 2m\pi$ "constructive"

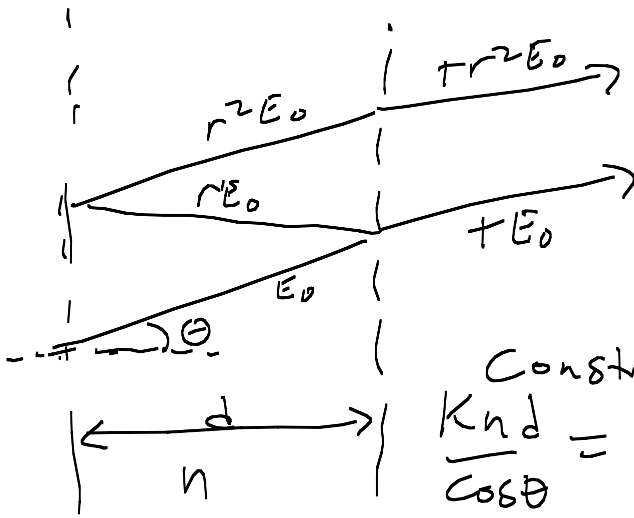
$$x = \frac{2m\pi L}{kd} = \frac{mL\lambda}{d}$$

## Example: Fabry – Perot Etalon




or




$$\alpha_1 - \alpha_2 = 2\frac{Knd}{\cos\theta}$$

$$E = tE_0\left\{2\cos\left(\frac{2\alpha_1 - 2Kn\frac{d}{\cos\theta}}{2} - \omega t\right) \cdot \cos\frac{Knd}{\cos\theta} + (1-r^2)\cos(\alpha_2 - \omega t)\right.$$

Constructive:

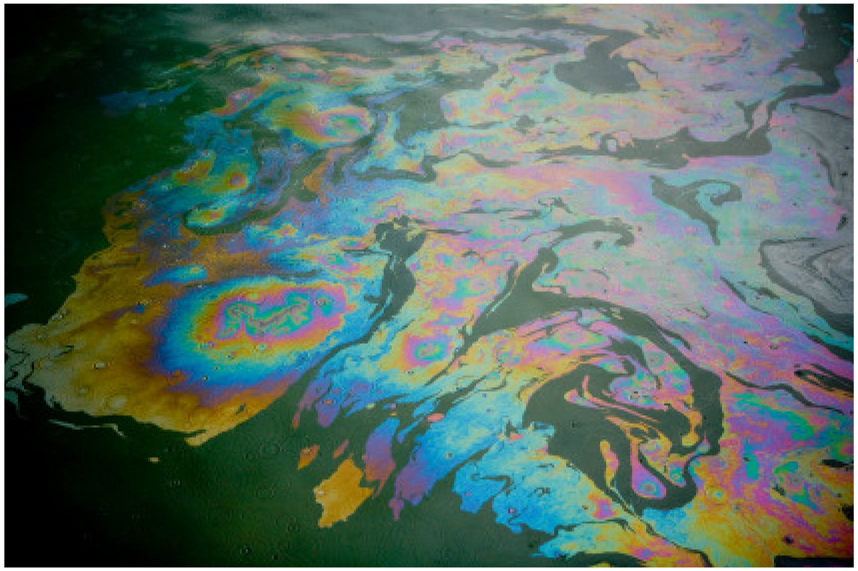
$$\frac{Knd}{\cos\theta} = 0, \pi, 2\pi, \dots m\pi \qquad d = \frac{m\pi\cos\theta}{Kn} = \frac{\lambda m}{2}\frac{\cos\theta}{n}$$

$$= \frac{\lambda_n}{2}, \lambda, \frac{3\lambda_n}{2}$$

destructive:

$$\frac{Knd}{\cos\theta} = \frac{\pi}{2}, \frac{3\pi}{2} \dots \left(\frac{2m+1}{2}\right)\pi \qquad d = \frac{\lambda_n}{4}, \frac{3\lambda_n}{4}, \dots$$

you have seen Fabry-Perot interference before:

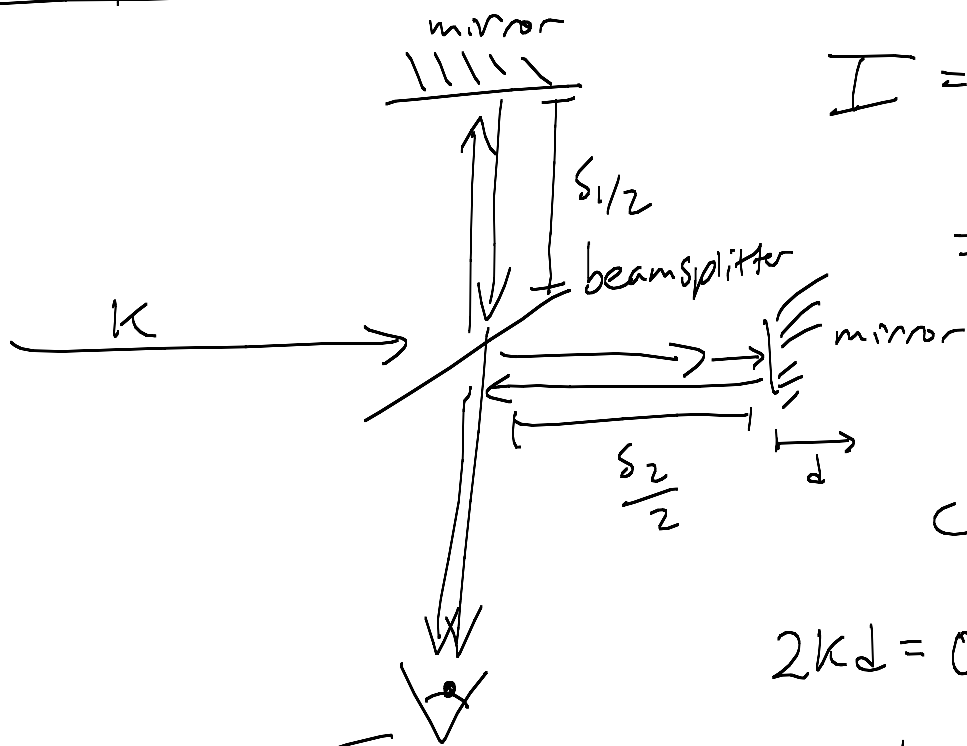


Oil slick

Soap bubble

Variations in thickness of the film changes the constructive/destructive conditions for different wavelengths. However, in the pics above, you see a particular color when it is constructively reflected, not transmitted.

# Example: Michelson Interferometer




$$I = \left(\frac{E_0}{2}\right)^2 + \left(\frac{E_0}{2}\right)^2 + 2\,\frac{E_0}{2}\,\frac{E_0}{2}\cos(\alpha_1 - \alpha_2)$$

$$= \frac{E_0^2}{2} + \frac{E_0^2}{2}\cos\big(k(s_1 - s_2)\big)$$

If $s_1 - s_2 = 2d$

Constructive interference for

$$2kd = 0,\ 2\pi,\ \dots\ 2m\pi$$

$2kd = 2\pi$

$d = \frac{\pi}{k} = \frac{\lambda}{2}$

destructive interference for

$$2kd = \pi,\ 3\pi,\ \dots\ (2m+1)\pi$$