Wave Equation

$$\nabla^2 \vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} \xrightarrow{\text{1-D}} \frac{\partial^2}{\partial z^2} E_{x,y} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} E_{x,y}$$

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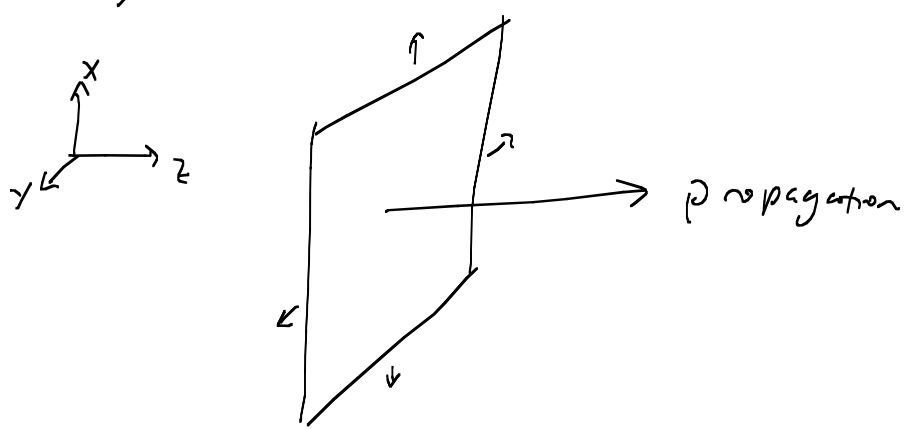
1-D solution: $E_{x,y} \propto e^{i(kz \pm \omega t)}$ "plane wave"



Surfaces of const. phase are infinite planes

How to find solutions of the wave eq. like confined beams we use in labs?

A more general solution

ansatz: $\vec{E}(x,y,z,t) = A(x,y,z)\,e^{i\omega t}$

$$\nabla^2 \vec{E} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E}$$

$$\nabla^2 A \cdot \cancel{e^{i\omega t}} = -\frac{\omega^2}{c^2} A\, \cancel{e^{i\omega t}} = -k^2 A$$

$$\nabla^2 A + k^2 A = 0$$ "Helmholtz eqn"

Simple Propagating Solution

Assume $\quad A(x,y,z) = u(x,y,z)\, e^{ikz} \qquad$ so that soln $\propto e^{i(kz-\omega t)}$

$$\nabla_T^2 u e^{ikz} + \frac{\partial^2}{\partial z^2}\left[u e^{ikz}\right] + k^2 u e^{ikz} = 0$$

x, y

$$\nabla_T^2 u e^{ikz} + \frac{\partial}{\partial z}\left[\frac{\partial u}{\partial z} e^{ikz} + iku e^{ikz}\right] + k^2 u e^{ikz} = 0$$

$$\nabla_T^2 u \cancel{e^{ikz}} + \cancel{\frac{\partial^2 u}{\partial z^2}} \cancel{e^{ikz}} + ik\frac{\partial u}{\partial z}\cancel{e^{ikz}} + ik\frac{du}{dz}\cancel{e^{ikz}} - \cancel{k^2 u e^{ikz}} + \cancel{k^2 u e^{ikz}} = 0$$

ignore for slowly-varying sol'n

$$\nabla_T^2 u + 2ik\frac{\partial u}{dz} = 0$$

Cylindrically symmetric soln:

$$\frac{\partial^2 u}{\partial r^2} = \frac{2k}{i}\frac{\partial u}{\partial z} \qquad \left(r = \sqrt{x^2 + y^2} \right)$$

If $t \equiv z$, $-\frac{\hbar}{2m} \equiv 1$, $\hbar \equiv 2k$, This is mathematically equivalent to time-dep. Schroedinger eqn. for free particle!

Solution via Fourier Transform:

$$u(r,z) = \int_{-\infty}^{\infty} \tilde{u}(k_r, z)\, e^{ik_r r}\, dk_r \qquad \left(\text{neglecting } \frac{1}{\sqrt{2\pi}} \text{ throughout}\right)$$

$$-k_r^2\, \tilde{u} = \frac{2k}{i}\frac{\partial \tilde{u}}{dz} \longrightarrow \tilde{u}(k_r, z) = A(k_r)\, e^{-i\frac{k_r^2}{2k}z}$$

Envelope function

$$u(r,z) = \int_{-\infty}^{\infty} A(k_r)\, e^{-i\frac{k_r^2}{2k}z}\, e^{ik_r r}\, dk_r$$

$A(k_r)$ determined by "initial conditions". If $u(r, z=0) = e^{-ax^2}$,

$A(k_r) \propto e^{-\frac{k_r^2}{4a}}$ (FT of gaussian is gaussian)

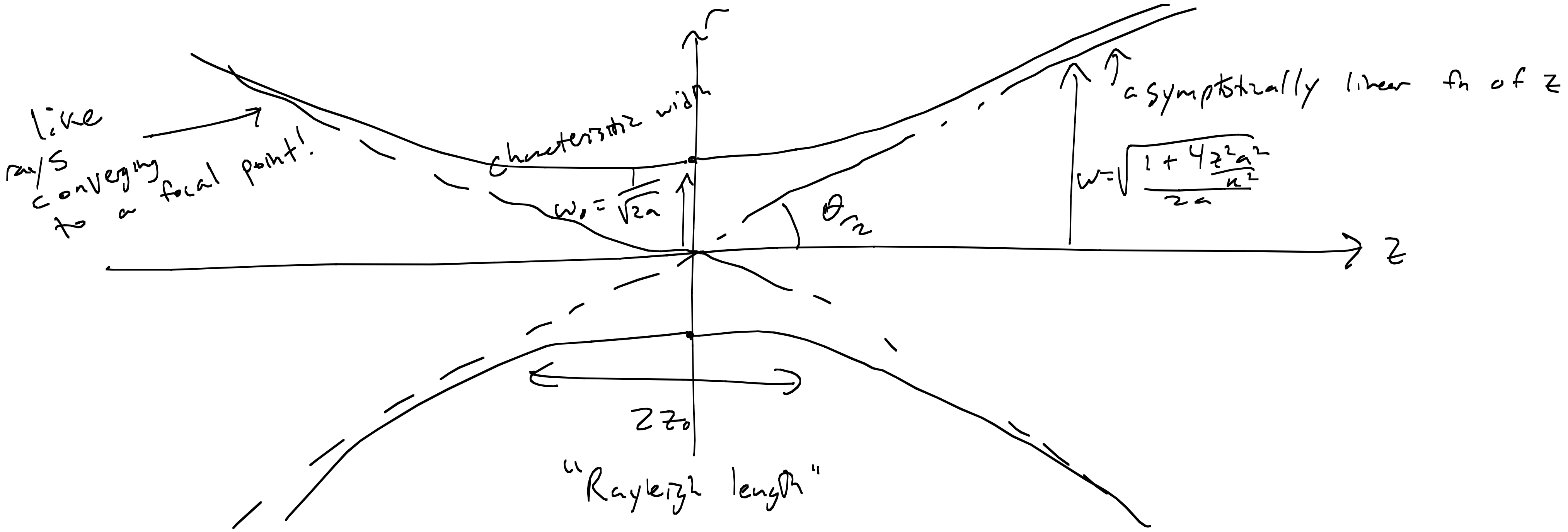
$$u(r,z) = \int_{-\infty}^{\infty} e^{-\frac{k_r^2}{4a}}\, e^{-i\frac{k_r^2}{2k}z}\, e^{ik_r r}\, dk_r$$

$$= \int_{-\infty}^{\infty} e^{-\left(\frac{1}{4a}+\frac{iz}{2k}\right)\left(k_r^2 - \frac{ik_r r}{\frac{1}{4a}+\frac{iz}{2k}}\right)} dk_r$$

$$= \int_{-\infty}^{\infty} \underbrace{e^{-\left(\frac{1}{4a}+\frac{iz}{2k}\right)\left(k_r - \frac{ir}{2\left(\frac{1}{4a}+\frac{iz}{2k}\right)}\right)^2}}_{\text{gaussian}}\; \underbrace{e^{-\frac{r^2}{4\left(\frac{1}{4a}+\frac{iz}{2k}\right)}}}_{\text{no } k_r!}\, dk_r$$

$$= \sqrt{\frac{\pi}{\frac{1}{4a} + \frac{iz}{2k}}}\; e^{-\frac{r^2}{\frac{1}{a} + \frac{2iz}{k}}}$$

$$I \propto |E|^2 \propto e^{-\frac{r^2}{\frac{1}{a} + \frac{2iz}{k}}}\, e^{-\frac{r^2}{\frac{1}{a} - \frac{2iz}{k}}} = e^{-r^2\left[\frac{2/a}{\frac{1}{a^2} + \frac{4z^2}{k^2}}\right]} = e^{-r^2\left[\frac{2a}{1 + \frac{4z^2a^2}{k^2}}\right]}$$



focal point

wave nature of light does not allow it to be confined to a point!

$$\frac{\theta}{2} \sim \frac{w_0}{z_0} = \frac{1}{\sqrt{2a z_0}}$$

lengthscale determined by unitless parameter

$$\frac{4 z_0^2 a^2}{k^2} \sim \frac{2a}{k^2 \left(\frac{\theta}{2}\right)^2} = \frac{8a}{k^2 \theta^2} = \frac{\frac{4}{w_0^2}}{k^2 \theta^2} \sim 1$$

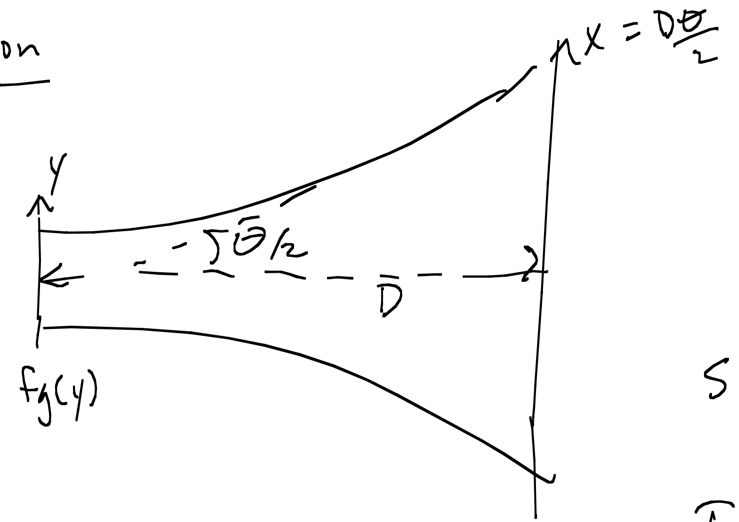
so $w_0 \sim \frac{2}{k\theta} = \frac{\lambda}{\pi\theta}$

$$\left(z_0 = \frac{2w_0}{\theta} = \frac{2\lambda}{\pi\theta^2} \right)$$

So for small focus, small λ and large θ

→ why "Blue-ray" DVD can have higher resolution since information density higher!

Diffraction



Consider the beam profile at $z=0$ as an "aperture":

$$f_g(y) = e^{-\frac{y^2}{w_0^2}} \qquad FT(f_g(y)) = e^{-\frac{k'^2 w_0^2}{4}}$$

Subst $k' \to \frac{kx}{D}$ $\qquad E(x) = e^{-\frac{k^2 x^2 w_0^2}{4D^2}}$

$$I(x) \propto e^{-\frac{k^2 x^2 w_0^2}{2D^2}}$$

lengthscale $\frac{\cancel{D}}{k w_0} = \frac{\cancel{D}\theta}{2}$

$$w_0 = \frac{2}{k\theta} = \frac{\lambda}{\pi\theta}$$ c.f. above!

So divergence of gaussian is due to diffraction!