Integral Form of Maxwell's Equations

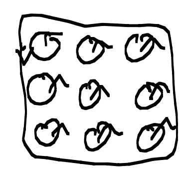
$$\oiint \vec{E}\cdot d\vec{S} = \frac{Q}{\varepsilon_0}$$ "Gauss' Law"

$$\oint \vec{E}\cdot d\vec{l} = -\frac{d\Phi_B}{dt}$$ "Faraday's Law"

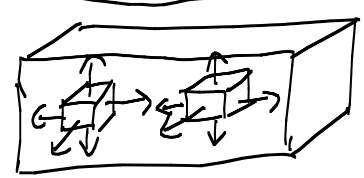
$$\oiint \vec{B}\cdot d\vec{S} = 0$$

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I + \mu_0\varepsilon_0 \frac{d\Phi_E}{dt}$$ "Ampere's Law"

"Stokes' Thm" $$\oint \vec{A}\cdot d\vec{l} = \iint \vec{\nabla}\times\vec{A}\cdot d\vec{S}$$



"divergence Thm" $$\oiint \vec{A}\cdot d\vec{S} = \iiint \vec{\nabla}\cdot\vec{A}\, dV$$



Maxwell's Eqns in Differential form

$$\oiint \vec{E}\cdot d\vec{S} = \iiint \vec{\nabla}\cdot\vec{E}\,dV = \frac{Q}{\varepsilon_0} = \frac{\iiint \rho\, dV}{\varepsilon_0} \Rightarrow \boxed{\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}}$$

$$\oint \vec{E}\cdot d\vec{l} = \iint \vec{\nabla}\times\vec{E}\cdot d\vec{S} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}\iint \vec{B}\cdot d\vec{S} \Rightarrow \boxed{\vec{\nabla}\times\vec{E} = -\frac{d\vec{B}}{dt}}$$

$$\oiint \vec{B}\cdot d\vec{S} = \iiint \vec{\nabla}\cdot\vec{B}\,dV = 0 \Rightarrow \boxed{\vec{\nabla}\cdot\vec{B} = 0}$$

$$\oint \vec{B}\cdot d\vec{l} = \iint \vec{\nabla}\times\vec{B}\cdot d\vec{S} = \mu_0 I + \mu_0\varepsilon_0\frac{d\Phi_E}{dt} = \iint\left[\mu_0\vec{J} + \mu_0\varepsilon_0\frac{d\vec{E}}{dt}\right]\cdot d\vec{S}$$

$$\Rightarrow \boxed{\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{d\vec{E}}{dt}}$$

Wave Equation

$\vec{\nabla}\times$ (Faraday's Law):

$$\vec{\nabla}\times\vec{\nabla}\times\vec{E} = \vec{\nabla}\times\left(-\frac{d\vec{B}}{dt}\right) = -\frac{d}{dt}\left(\vec{\nabla}\times\vec{B}\right) = -\frac{d}{dt}\left(\cancel{\mu_0\vec{J}}^{\,0} + \mu_0\varepsilon_0\frac{d\vec{E}}{dt}\right)$$

in free space

$$\vec{\nabla}\left(\cancel{\vec{\nabla}\cdot\vec{E}}^{\,0}\right) - \nabla^2\vec{E} = -\mu_0\varepsilon_0\frac{d^2\vec{E}}{dt^2}$$

in free space

$$\nabla^2\vec{E} = \mu_0\varepsilon_0\frac{d^2\vec{E}}{dt^2}$$

$$\mu_0\varepsilon_0 \equiv \frac{1}{c^2} \qquad [\mu_0\varepsilon_0] = \frac{1}{\text{Velocity}^2}$$

$$\nabla^2\vec{E} = \frac{1}{c^2}\frac{\partial^2\vec{E}}{dt^2}$$

"Wave equation"

Note: in materials w/ $\varepsilon \neq \varepsilon_0$ and/or $\mu \neq \mu_0$, Velocity $\equiv \frac{c}{n}$ where n is "index of refraction"

1-D $\left(\nabla^2 \to \frac{\partial^2}{\partial z^2}\right)$ solution

Any $F(z \pm ct)$! "D'Alembert" solution

"Simple" example in 1-D:

$$E = E_0 e^{i(kz \pm \omega t)} = E_0 e^{ik\left(z \pm \frac{\omega}{k} t\right)}$$ "plane wave"

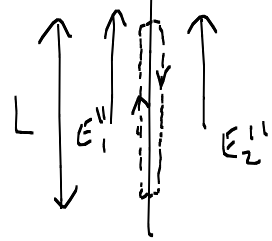
$\frac{\omega}{k} = \frac{c}{n}$ $\quad$ $k \to$ "wavenumber" $\quad$ $k = \frac{2\pi}{\lambda}$ ← "wavelength"

Boundary Conditions for Maxwell's eqns

E-field

interface

$\varepsilon_1, \mu_1 \quad \varepsilon_2, \mu_2$



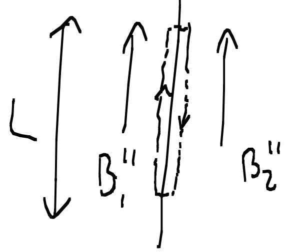
$$\oint \vec{E}\cdot d\vec{l} = -\cancel{\frac{d\Phi_B}{dt}} \to 0$$ "Faraday's Law"

$$E_1^{\parallel} L - E_2^{\parallel} L = 0$$

$$E_1^{\parallel} = E_2^{\parallel}$$

B-field

$\varepsilon_1, \mu_1 \quad \varepsilon_2, \mu_2$



$$\oint \vec{B}\cdot d\vec{l} = \mu I + \mu\varepsilon \cancel{\frac{d\Phi_E}{dt}} \to 0$$ "Ampere's Law"

$$\oint \frac{\vec{B}}{\mu}\cdot d\vec{l} = 0$$

$$\frac{B_1^{\parallel}}{\mu_1} L - \frac{B_2^{\parallel}}{\mu_2} L = 0$$

$$\frac{B_1^{\parallel}}{\mu_1} = \frac{B_2^{\parallel}}{\mu_2} \qquad \left(H_1^{\parallel} = H_2^{\parallel}\right)$$

Plane wave $\vec{B}$ Components from $\vec{E}$

$$\vec{E} = E_x e^{i(kz-\omega t)}\hat{x} \qquad (E_y = E_z = 0)$$

Faraday's Law: $\vec{\nabla}\times\vec{E} = -\frac{d\vec{B}}{dt}$

$$\vec{\nabla}\times\vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ E_x e^{i(kz-\omega t)} & 0 & 0 \end{vmatrix} = ikE_x e^{i(kz-\omega t)}\hat{y}$$

$$\vec{B} = (B_x\hat{x} + B_y\hat{y} + B_z\hat{z})e^{i(kz-\omega t)}$$

$$-\frac{d\vec{B}}{dt} = -\left(-i\omega(B_x\hat{x} + B_y\hat{y} + B_z\hat{z})e^{i(kz-\omega t)}\right)$$

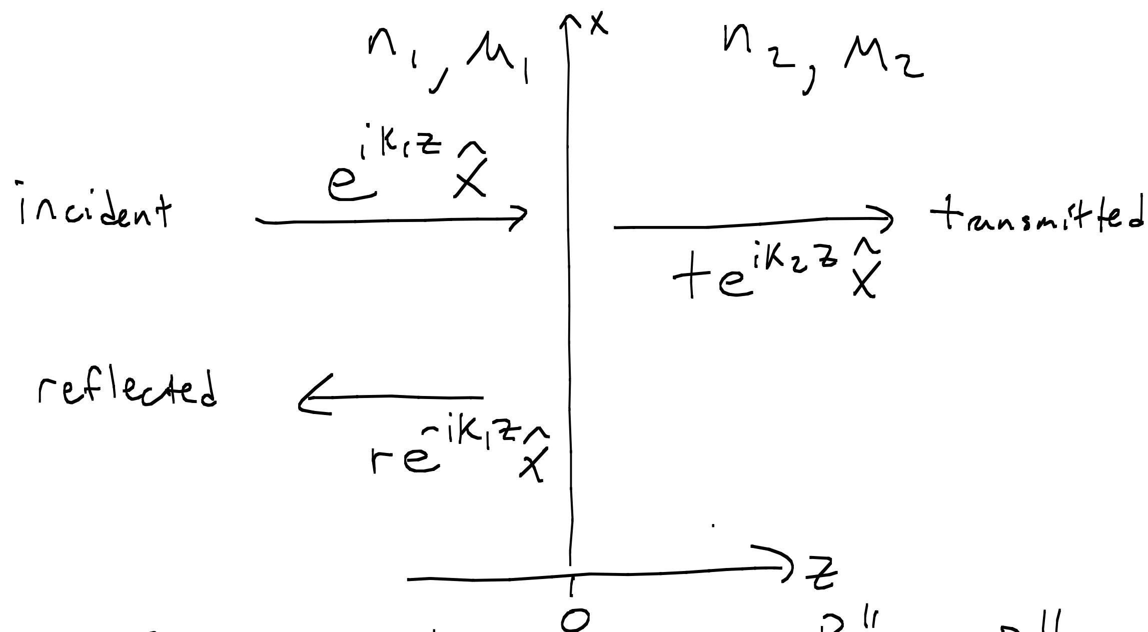
So $\vec{\nabla}\times\vec{E} = -\frac{d\vec{B}}{dt}$ implies $ikE_x e^{i(kz-\omega t)}\hat{y} = i\omega B_y e^{i(kz-\omega t)}\hat{y}$

Therefore, $B_y = \frac{k}{\omega}E_x = \frac{E_x}{v} = \frac{nE_x}{c} \qquad (B_x = B_z = 0)$

Note: $\vec{E}\times\vec{B}$ is propagation direction $(\hat{x}\times\hat{y} = \hat{z})$!

Imposing Boundary Conditions

$\vec{E}$-field:



B.C.'s: (1) $E_1^{\parallel} = E_2^{\parallel}$ (2) $\dfrac{B_1^{\parallel}}{\mu_1} = \dfrac{B_2^{\parallel}}{\mu_2}$ $\left(B^{\parallel} = \dfrac{n}{c}E^{\parallel}\right)$

(1): $e^{ik_1 z} + re^{-ik_1 z}\Big|_{z=0} = te^{ik_2 z}\Big|_{z=0} \longrightarrow 1 + r = t$

$B^{\parallel} = \frac{E^{\parallel}}{v}$

(2): $\dfrac{n_1}{\mu_1 c}e^{ik_1 z} - \dfrac{n_1}{\mu_1 c} re^{-ik_1 z}\Big|_{z=0} = \dfrac{n_2}{\mu_2 c}te^{ik_2 z}\Big|_{z=0} \longrightarrow \dfrac{n_1}{\mu_1} - r\dfrac{n_1}{\mu_1} = \dfrac{n_2}{\mu_2}t$

$$\frac{n_1}{\mu_1} - r\frac{n_1}{\mu_1} = \frac{n_2}{\mu_2}(1+r)$$

$$\frac{n_1}{\mu_1} = \frac{n_2}{\mu_2} + r\left(\frac{n_1}{\mu_1} + \frac{n_2}{\mu_2}\right)$$

$$r = \frac{\frac{n_1}{\mu_1} - \frac{n_2}{\mu_2}}{\frac{n_1}{\mu_1} + \frac{n_2}{\mu_2}}$$

Reflection Coefficient:

$$R = |r|^2 = \left(\frac{\frac{n_1}{\mu_1} - \frac{n_2}{\mu_2}}{\frac{n_1}{\mu_1} + \frac{n_2}{\mu_2}}\right)^2$$

When $\mu_1 = \mu_2$,

$$R = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2$$

"transmission coef"

$$R + T = 1 \qquad T = 1 - R$$

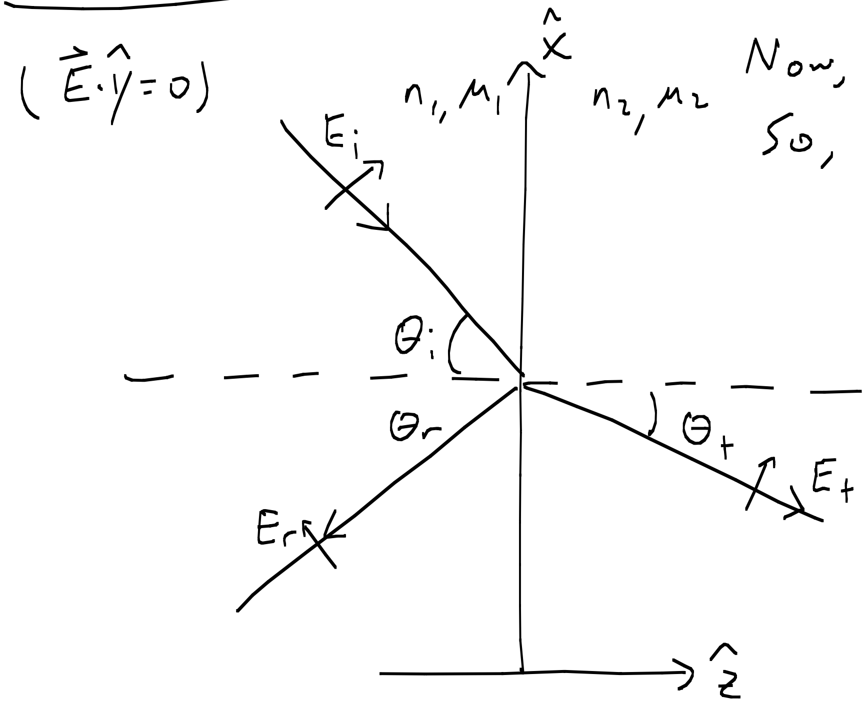
$$T = 1 - \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 = \frac{(n_1 + n_2)^2}{(n_1 + n_2)^2} - \frac{(n_1 - n_2)^2}{(n_1 + n_2)^2}$$

$$= \frac{n_1^2 + n_2^2 + 2n_1n_2 - (n_1^2 + n_2^2 - 2n_1n_2)}{(n_1 + n_2)^2}$$

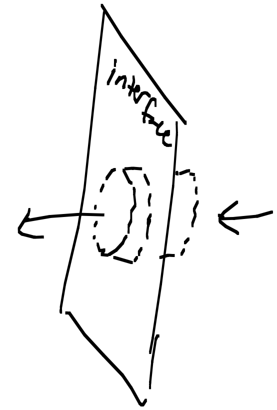
$$= \frac{4n_1n_2}{(n_1 + n_2)^2}$$

Reflection is consequence of field conservation at the interface!

What about oblique incidence?



Now, we have 4 unknowns $(E_r, \theta_r, E_t, \theta_t)$.
So, we need 2 more B.C.'s?:



$\oiint \vec{B}\cdot d\vec{s} = 0$ implies

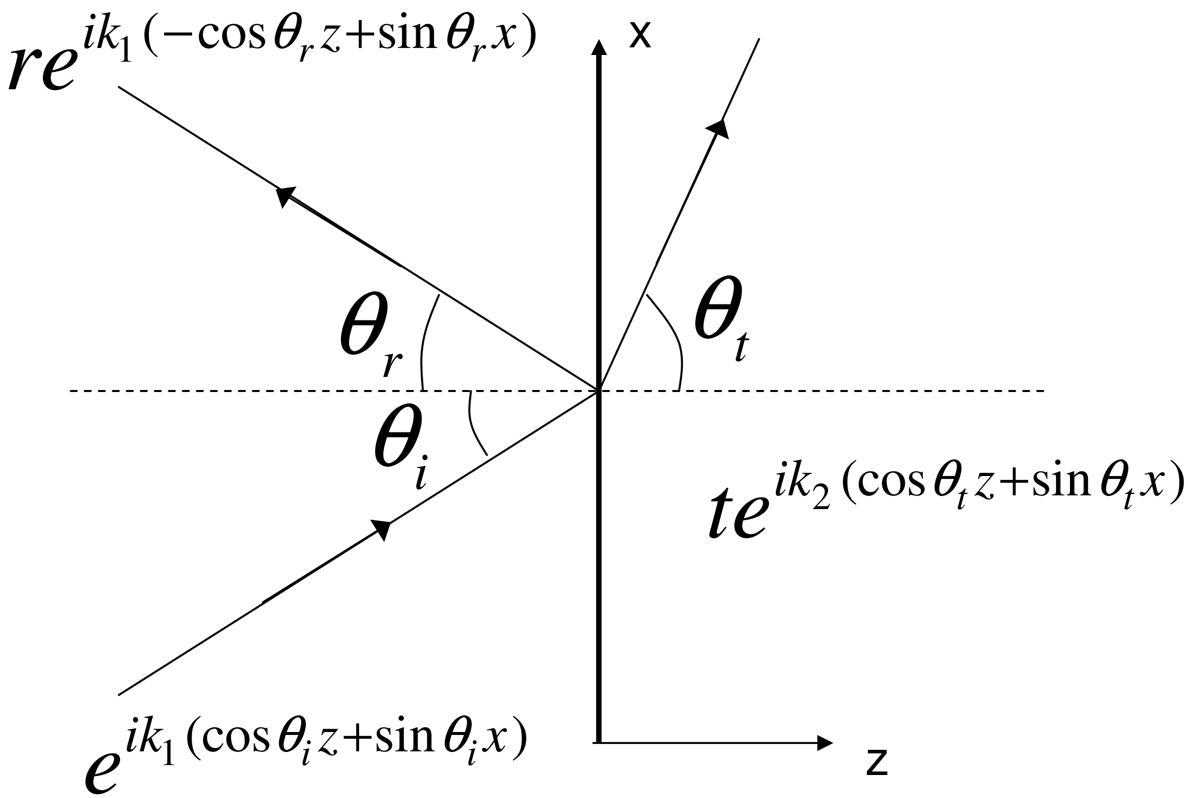
$$③\ B_1^\perp = B_2^\perp$$

and

$$\oiint \vec{E}\cdot d\vec{s} = \frac{\cancel{Q}^{\,0}}{\varepsilon} = 0$$ no free charge

implies

$$④\ \epsilon_1 E_1^\perp = \epsilon_2 E_2^\perp$$



BCs, e.g. $E_{\|}^1 = E_{\|}^2$:

$$\cos\theta_i\, e^{ik_1(\cos\theta_i z+\sin\theta_i x)}\Big|_{z=0} + r\cos\theta_r e^{ik_1(-\cos\theta_r z+\sin\theta_r x)}\Big|_{z=0} = t\cos\theta_t e^{ik_2(\cos\theta_t z+\sin\theta_t x)}\Big|_{z=0}$$

This is only true for all *x* if:

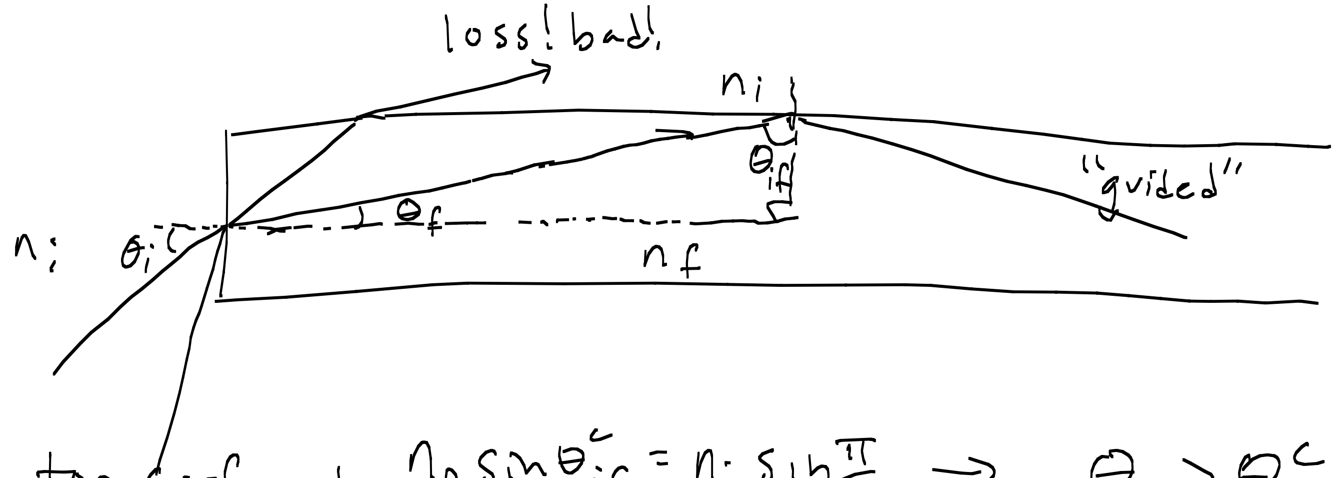
$$\frac{\omega}{k_1} = \frac{c}{n_1}$$

$$k_1 \sin\theta_i = k_1 \sin\theta_r = k_2 \sin\theta_t$$

$$k_1 = \frac{n_1 \omega}{c}$$

$$\theta_i = \theta_r \qquad n_1 \sin\theta_i = n_2 \sin\theta_t$$

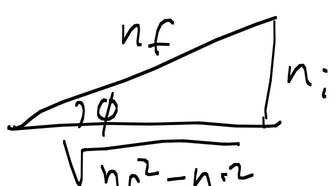
TIR application: fiber optics



top surface! (sides of fiber) $n_f \sin\theta_{if}^c = n_i \sin\frac{\pi}{2} \rightarrow \theta_{if} > \theta_{if}^c = \sin^{-1}\frac{n_i}{n_f}$

$$\sin\theta_f = \cos\theta_{if}$$

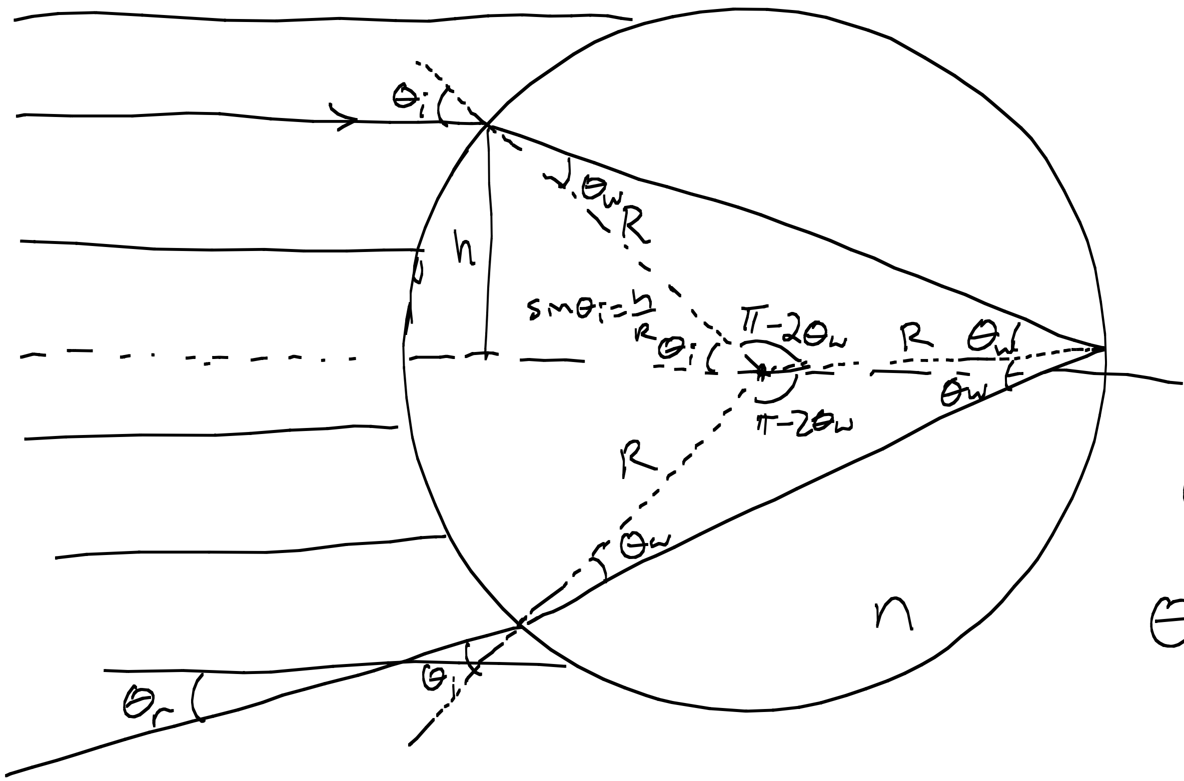
Input interface: $n_i \sin\theta_i^c = n_f \sin\theta_f^c = n_f \cos\theta_{if}^c = n_f \cos\left(\sin^{-1}\frac{n_i}{n_f}\right)$



$$n_i \sin\theta_i^c = n_f \frac{\sqrt{n_f^2-n_i^2}}{n_f}$$

$$\theta_i < \theta_i^c = \sin^{-1}\frac{\sqrt{n_f^2-n_i^2}}{n_i} = \sin^{-1}\sqrt{\left(\frac{n_f}{n_i}\right)^2-1}$$

Refraction and Reflection in a sphere



$$\theta_{tot} = 2(\pi - 2\theta_w) + 2\theta_i$$

$$\theta_r = 2\pi - \theta_{tot} = 4\theta_w - 2\theta_i$$

$$= 4\sin^{-1}\left(\frac{\sin\theta_i}{n}\right) - 2\theta_i$$

$$\sin\theta_i = n\sin\theta_w$$

$$\theta_w = \sin^{-1}\frac{\sin\theta_i}{n}$$

$$\theta_i = \sin^{-1}\frac{h}{R}$$

Caustics

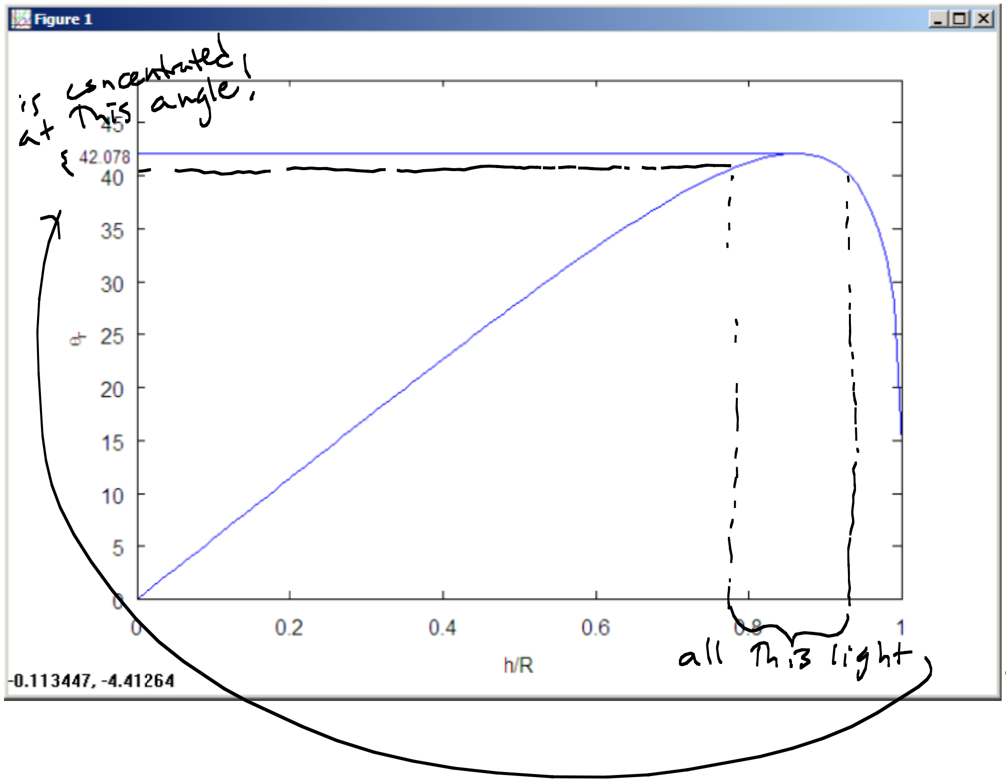


Figure 1
is concentrated at this angle!
42.078
0 5 10 15 20 25 30 35 40 45
0 0.2 0.4 0.6 0.8 1
h/R
all this light
-0.113447, -4.41264

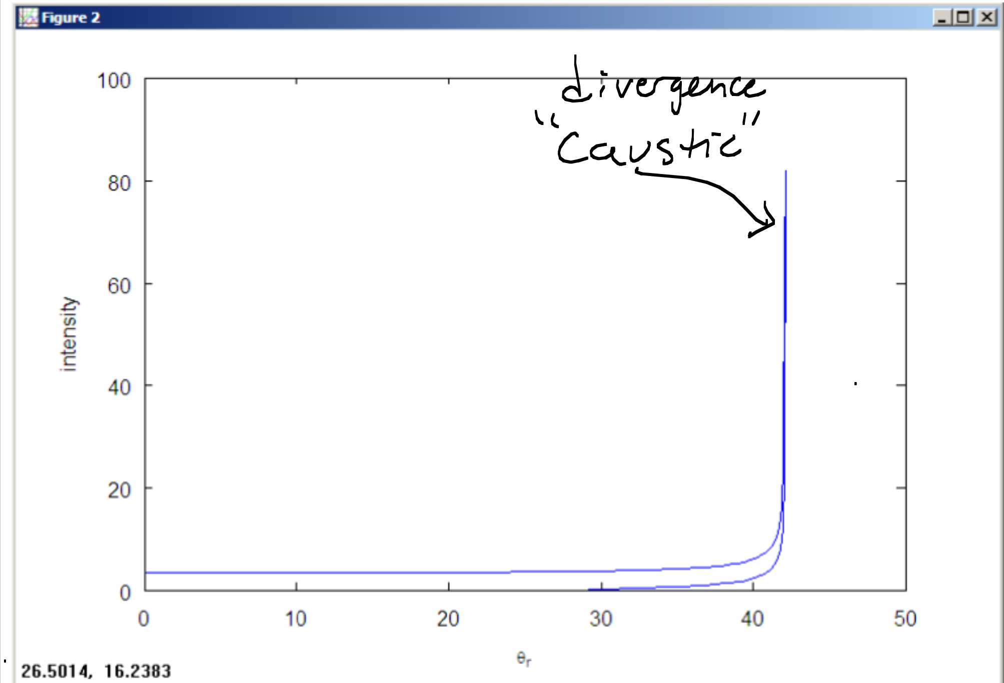
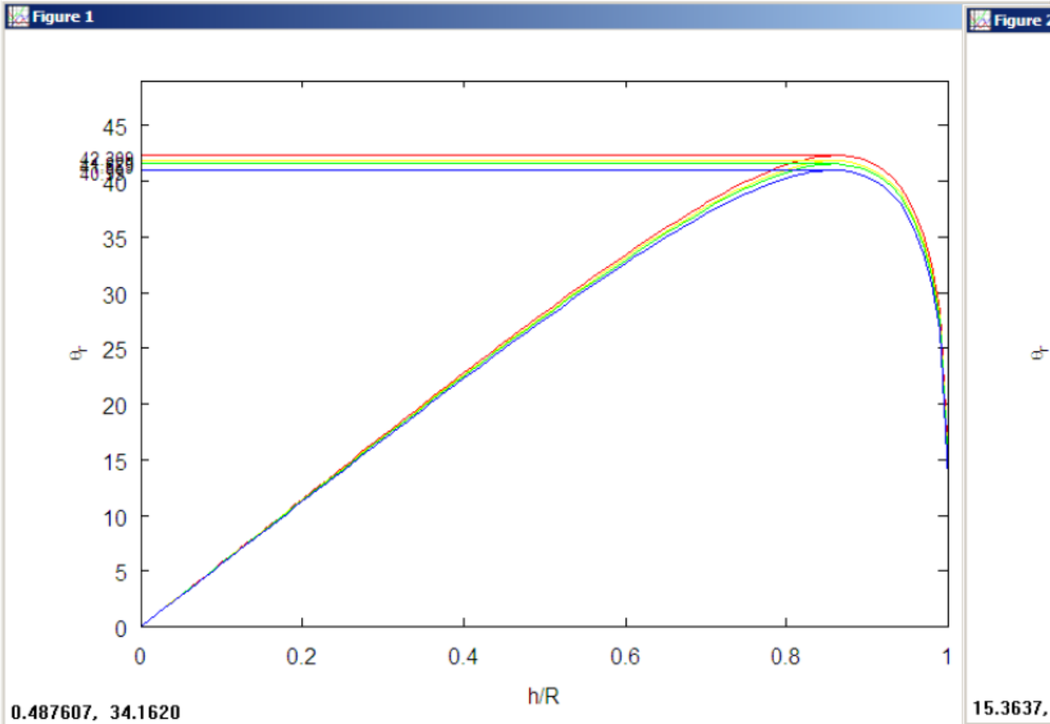
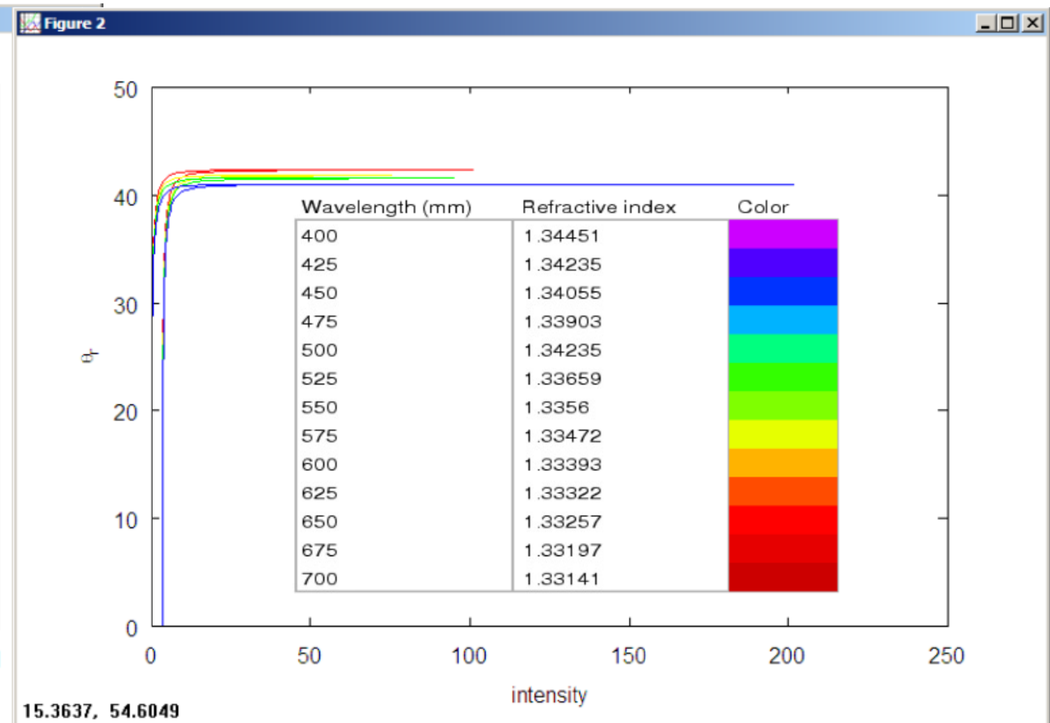


Figure 2
divergence
“caustic”
0 20 40 60 80 100
intensity
0 10 20 30 40 50
θr
26.5014, 16.2383

Rainbows





Wavelength (mm)	Refractive index	Color
400	1.34451	
425	1.34235	
450	1.34055	
475	1.33903	
500	1.34235	
525	1.33659	
550	1.3356	
575	1.33472	
600	1.33393	
625	1.33322	
650	1.33257	
675	1.33197	
700	1.33141	

Second rainbow from 2 internal reflections
→ colors reversed!

primary rainbow