Superposition of Waves

Wave eqn. $\nabla^2 \vec{E}_1 = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E}_1$

If $\vec{E}_2$ is also a solution, $\nabla^2 \vec{E}_2 = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E}_2$

Then $\vec{E} = \vec{E}_1 + \vec{E}_2$ is also:

$$\nabla^2(\vec{E}_1 + \vec{E}_2) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(\vec{E}_1 + \vec{E}_2)$$

linearity of operators gives

$$\nabla^2 \vec{E}_1 + \nabla^2 \vec{E}_2 = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E}_1 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E}_2$$

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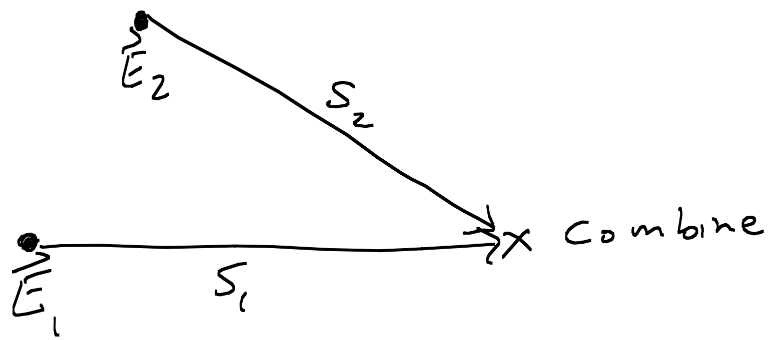
Superposition of 2 waves (same K and polarization)



$$|\vec{E}_1| = E_{01}\cos(\alpha_1 - \omega t) \qquad |\vec{E}_2| = E_{02}\cos(\alpha_2 - \omega t)$$

$$\alpha_1 = ks_1 + \phi_1 \qquad \alpha_2 = ks_2 + \phi_2$$

$$E_1 + E_2 = E_{01}\cos(\alpha_1 - \omega t) + E_{02}\cos(\alpha_2 - \omega t)$$

$$= E_{01}\cos(\alpha_1 - \omega t) + E_{01}\cos(\alpha_2 - \omega t) + (E_{02} - E_{01})\cos(\alpha_2 - \omega t)$$

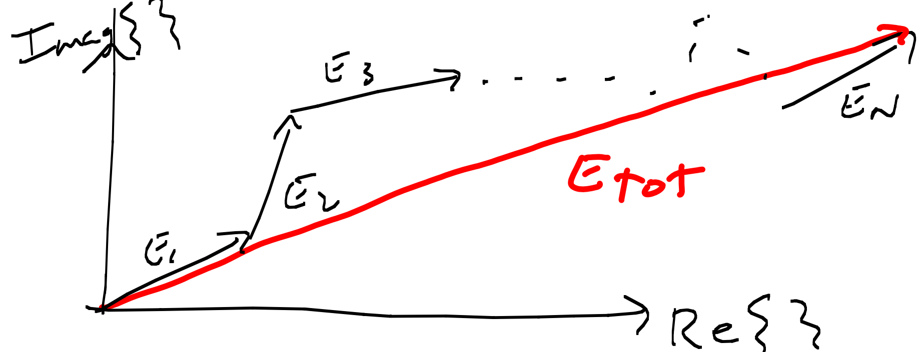
$$\boxed{\cos u + \cos v = 2\cos\frac{u+v}{2}\cos\frac{u-v}{2}}$$

$$= 2E_{01}\cos\left(\frac{\alpha_1 + \alpha_2 - 2\omega t}{2}\right)\cos\left(\frac{\alpha_1 - \alpha_2}{2}\right) + (E_{02} - E_{01})\cos(\alpha_2 - \omega t)$$

If $\alpha_1 - \alpha_2 = 0, 2\pi, \ldots m(2\pi)$ $\quad m = 0, \pm 1, \pm 2 \ldots$ "constructive" interference

$= \pi, 3\pi, \ldots (2m+1)\pi$ "destructive"

Superposition of N waves: phasors



$$E_i = E_{0i} e^{i(\alpha_i - \omega t)}$$

$$\vec{E}_{tot} = \left(\sum_i^N E_{0i} \sin\alpha_i\right) i + \left(\sum_i^N E_{0i} \cos\alpha_i\right)$$

$$I = |E_{tot}|^2 = \left(\sum_i^N E_{0i} \sin\alpha_i\right)^2 + \left(\sum_i^N E_{0i} \cos\alpha_i\right)^2$$

$$= \sum_i^N E_{0i}^2 \sin^2\alpha_i + 2\sum_{j>i}^N \sum_{i=1}^N E_{0i} E_{0j} \sin\alpha_i \sin\alpha_j$$

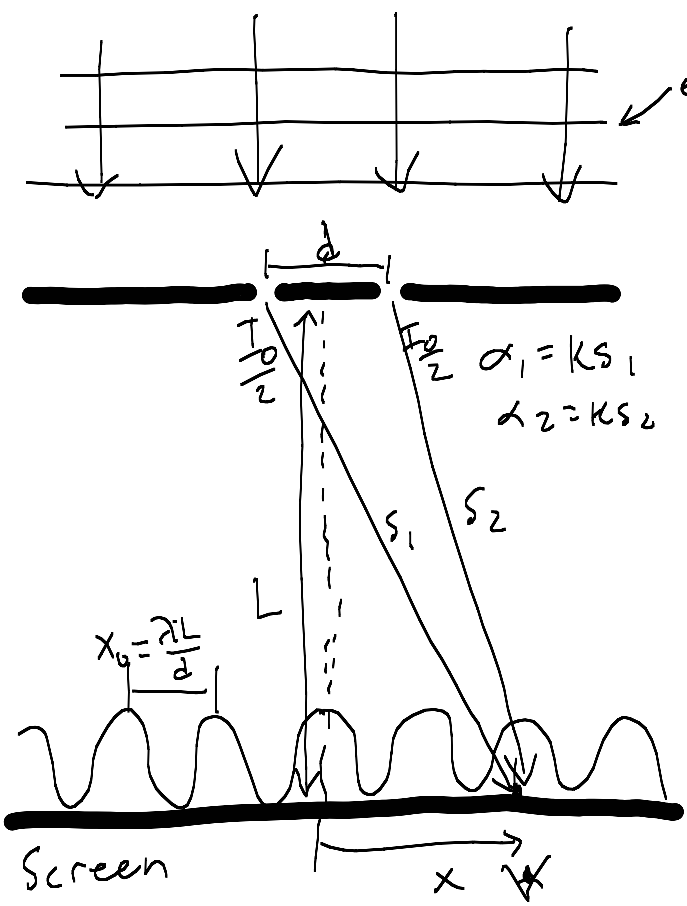
$$+ \sum_i^N E_{0i}^2 \cos^2\alpha_i + 2\sum_{j>i}^N \sum_{i=1}^N E_{0i} E_{0j} \cos\alpha_i \cos\alpha_j$$

$$\boxed{\begin{array}{l} \text{e.g.} \\ (a+b)^2 \\ a^2+b^2+ab+ba \\ a^2+b^2+2ab \end{array}}$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

$$= \sum_i^N E_{0i}^2 + 2\sum_{j>i}^N \sum_{i=1}^N E_{0i} E_{0j} \cos(\alpha_j - \alpha_i)$$

Example: Young's Two-slit experiment



$$I(x) = \frac{I_0}{2} + \frac{I_0}{2} + 2\sqrt{\frac{I_0}{2}\frac{I_0}{2}}\cos(\alpha_1 - \alpha_2)$$

$$= I_0 + I_0 \cos(\alpha_1 - \alpha_2)$$

$$\alpha_1 - \alpha_2 = k(s_1 - s_2) = k\left(\sqrt{L^2 + (x+\tfrac{d}{2})^2} - \sqrt{L^2 + (x - \tfrac{d}{2})^2}\right)$$

$$= kL\left(\sqrt{1 + \frac{(x+\frac{d}{2})^2}{L^2}} - \sqrt{1 + \frac{(x-\frac{d}{2})^2}{L^2}}\right)$$

$$\sim kL\left(1 + \frac{(x+\frac{d}{2})^2}{2L^2} - \left(1 + \frac{(x-\frac{d}{2})^2}{2L^2}\right)\right)$$

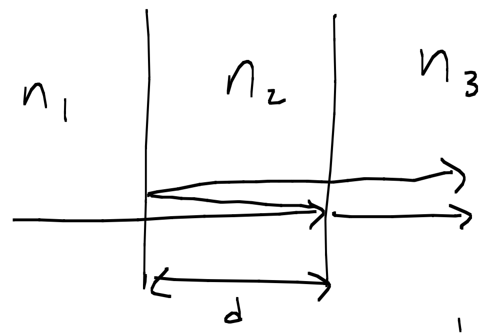
$$= kL\left(\frac{x^2 + xd + \frac{d^2}{4} - \left(x^2 - xd + \frac{d^2}{4}\right)}{2L^2}\right)$$

$$= kL\left(\frac{2xd}{2L^2}\right) = \frac{kxd}{L}$$

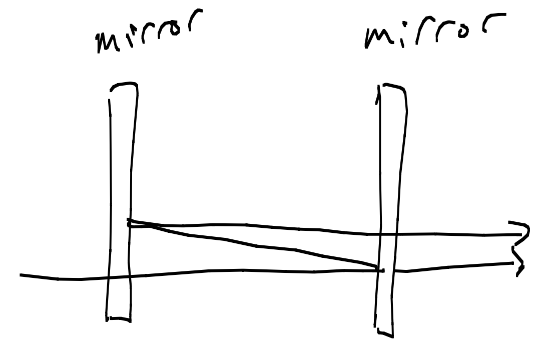
$$I(x) = I_0\left(1 + \cos\frac{kd}{L}x\right)$$

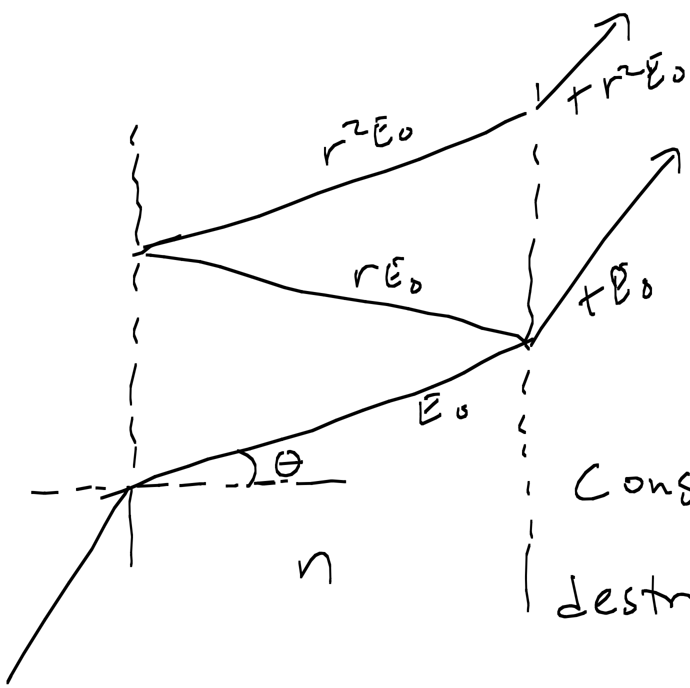
$$\frac{kd}{L}x_0 = 2\pi \qquad x_0 = \frac{2\pi L}{kd} = \frac{\lambda L}{d}$$

Example: Fabry-Perot Etalon



or





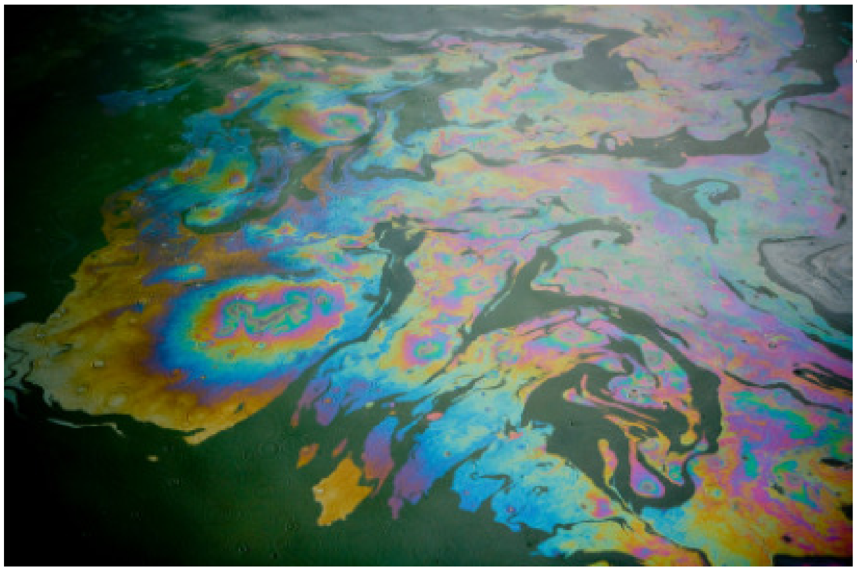
$$\alpha_1 - \alpha_2 = 2\frac{knd}{\cos\theta}$$

$$E = tE_0\left[2\cos\left(\frac{2\alpha_1 - 2kn\frac{d}{\cos\theta}}{2} - \omega t\right)\cos\frac{knd}{\cos\theta} + (1-r^2)\cos(\alpha_2 - \omega t)\right]$$

Constructive $\frac{knd}{\cos\theta} = 0, \pi, 2\pi, \ldots \; m\pi$ $\quad d = \frac{m\pi\cos\theta}{kn} \left(= \frac{\lambda}{2}, \lambda, \ldots\right)$

destructive $\frac{knd}{\cos\theta} = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots \frac{(2m+1)}{2}\pi$ $\quad d = \frac{2m+1}{2}\frac{\cos\theta}{kn} \left(= \frac{\lambda}{4}, \frac{3\lambda}{4}, \ldots\right)$

You have seen Fabry-Perot interference before:

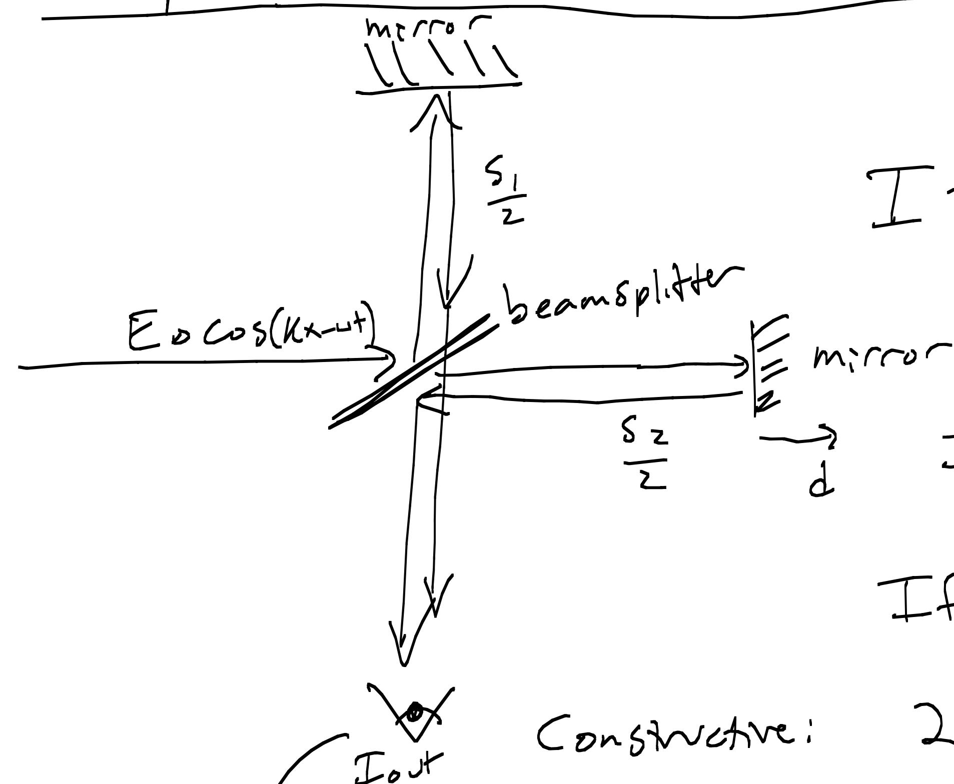


Oil slick

Soap bubble

Variations in thickness of the film changes the constructive/destructive conditions for different wavelengths. However, in the pics above, you see a particular color when it is constructively reflected, not transmitted.

Example: Michelson Interferometer



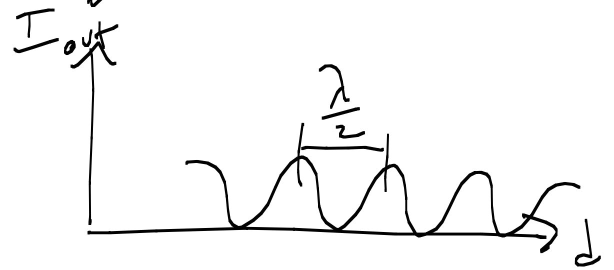
$$I = \left(\frac{E_0}{2}\right)^2 + \left(\frac{E_0}{2}\right)^2 + 2\,\frac{E_0}{2}\frac{E_0}{2}\cos[k(s_1-s_2)]$$

$$= \frac{E_0^2}{2} + \frac{E_0^2}{2}\cos[k(s_1-s_2)]$$

If $s_1 - s_2 = 2d$

Constructive: $2kd = 0, 2\pi, \dots 2m\pi$

$d = \frac{m\pi}{k} = \frac{m\lambda}{2}$ $\quad m = 0, \pm 1, \pm 2, \dots$



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