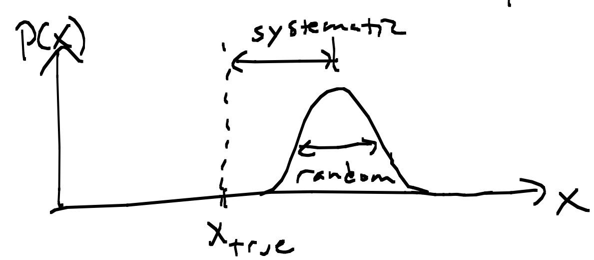
Measurement Error

Systematic errors $\longrightarrow$ accuracy

random errors $\longrightarrow$ precision



Properties of probability distribution functions:

$$\int_{-\infty}^{\infty} P(x)\,dx = 1$$

"moments":

1st: $$\frac{\int_{-\infty}^{\infty} P(x)\,x\,dx}{\int_{-\infty}^{\infty} P(x)\,dx} = \bar{x}$$ "mean"

2nd: $$\frac{\int_{-\infty}^{\infty} P(x)\,(x-\bar{x})^2\,dx}{\int_{-\infty}^{\infty} P(x)\,dx} = \sigma^2$$ "variance"

σ "standard deviation"

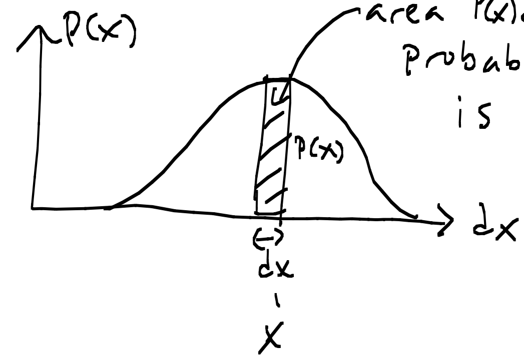
3rd: $$\frac{\int_{-\infty}^{\infty} P(x)\,(x-\bar{x})^3\,dx}{\sigma^3 \int_{-\infty}^{\infty} P(x)\,dx}$$ "skew" (asymmetry)

4th: $$\frac{\int_{-\infty}^{\infty} P(x)\,(x-\bar{x})^4\,dx}{\sigma^4 \int_{-\infty}^{\infty} P(x)\,dx} - 3$$ "Kurtosis" (peakedness)

Sampling

"Mean":

$$\frac{\int_{-\infty}^{\infty} x \overbrace{P(x)\,dx}^{\text{included in measurement!}}}{\int_{-\infty}^{\infty} P(x)\,dx}$$



area P(x)dx is relative probability that x is chosen

Sampling $\Longrightarrow$

$$\frac{\sum_i^N x_i}{\sum_i^N (1)} = \frac{\sum_i^N x_i}{N}$$

"Variance":

$$\frac{\int_{-\infty}^{\infty} (x-\bar{x})^2 P(x)\,dx}{\int_{-\infty}^{\infty} P(x)\,dx} \Longrightarrow \frac{\sum_i^N (x_i-\bar{x})^2}{N-\underbrace{1}}$$

$\bar{x}$ not independent of x_i

"mode": most common value(s)

"median": middle value $\rightarrow$ half larger, half smaller

} other ways to characterize lists of values

Combining measurements

Lacking any other info, we assume that probability distribution function is

$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\bar{x})^2}{2\sigma^2}}$$

This is convenient: Gaussian is a simple distribution w/o skew or kurtosis!

If two measurements of the same variable are made, each with result x_A and x_B

$$P(x_A) = \frac{1}{\sigma_A\sqrt{2\pi}} e^{-\frac{(x_A-\bar{x})^2}{2\sigma_A^2}} \qquad P(x_B) = \frac{1}{\sigma_B\sqrt{2\pi}} e^{-\frac{(x_B-\bar{x})^2}{2\sigma_B^2}}$$

$$P(x_A, x_B) = P(x_A)P(x_B) \propto e^{-\frac{1}{2}\left[\frac{(x_A-\bar{x})^2}{\sigma_A^2} + \frac{(x_B-\bar{x})^2}{\sigma_B^2}\right]}$$

We assume the maximum probability event has occurred, so maximize $P(x_A, x_B)$ w.r.t $\bar{x}$. This is equivalent to minimizing the exponent:

$$\frac{d}{d\bar{x}}\left[\frac{(x_A-\bar{x})^2}{\sigma_A^2} + \frac{(x_B-\bar{x})^2}{\sigma_B^2}\right] = 0$$

$$\frac{-2(x_A-\bar{x})}{\sigma_A^2} + \frac{-2(x_B-\bar{x})}{\sigma_B^2} = 0$$

If $\quad w_A \equiv \frac{1}{\sigma_A^2}, \quad w_B \equiv \frac{1}{\sigma_B^2},$

$$w_A(x_A - \bar{x}) + w_B(x_B - \bar{x}) = 0$$

$$w_A x_A + w_B x_B - \bar{x}(w_A + w_B) = 0$$

$$\bar{x} = \frac{w_A x_A + w_B x_B}{w_A + w_B}$$

$\xrightarrow{\text{in general,}}$

$$\bar{x} = \frac{\sum_i w_i x_i}{\sum_i w_i}$$

So measurements with large σ count less toward $\bar{x}$!

Standard deviation of the mean

If you make N measurements, what is the uncertainty in $\bar{x}$?

$$\prod_i^N P_i(x_i) = e^{-\frac{(x-\bar{x})^2}{2\sigma_1^2}} e^{-\frac{(x-\bar{x})^2}{2\sigma_2^2}} e^{-\frac{(x-\bar{x})^2}{2\sigma_3^2}} \cdots e^{-\frac{(x-\bar{x})^2}{2\sigma_N^2}}$$

$$= e^{-\sum_i \frac{1}{\sigma_i^2} \frac{(x-\bar{x})^2}{2}} \longrightarrow \frac{1}{\sigma_{\bar{x}}^2} = \sum_i^N \frac{1}{\sigma_i^2}$$

So $\sigma_{\bar{x}}$ always decreases with more measurements (even ones w/ large σ)

If all σ_i^2 are same, $\frac{1}{\sigma_{\bar{x}}^2} = \frac{N}{\sigma^2}$, so $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{N}}$

Propagation of errors

you want to calculate $f(x,y)$ using sampled data x_i and y_i, with uncertainties σ_x and σ_y. What is σ_f?

$$\sigma_f^2 = \frac{1}{N} \sum_i^N \left(f(x_i, y_i) - f(\bar{x}, \bar{y}) \right)^2$$

If we taylor expand $f(x,y)$ to first order,

$$f(x_i, y_i) \sim f(\bar{x}, \bar{y}) + \frac{df}{dx}(x_i - \bar{x}) + \frac{df}{dy}(y_i - \bar{y})$$

So

$$\sigma_f^2 = \frac{1}{N} \sum_i^N \left(\cancel{f(\bar{x}, \bar{y})} + \frac{df}{dx}(x_i - \bar{x}) + \frac{df}{dy}(y_i - \bar{y}) - \cancel{f(\bar{x}, \bar{y})} \right)^2$$

$$\sigma_f^2 = \frac{1}{N}\sum_i^N \left[\left(\frac{df}{dx}\right)^2 (x_i-\bar{x})^2 + \left(\frac{df}{dy}\right)^2 (y_i-\bar{y})^2 + 2\frac{df}{dx}\frac{df}{dy}(x_i-\bar{x})(y_i-\bar{y})\right]$$

$$= \frac{1}{N}\left[\left(\frac{df}{dx}\right)^2 \underbrace{\sum_i (x_i-\bar{x})^2}_{\to\sigma_x^2} + \left(\frac{df}{dy}\right)^2 \underbrace{\sum_i (y_i-\bar{y})^2}_{\to\sigma_y^2}\right] + 2\frac{df}{dx}\frac{df}{dy}\underbrace{\sigma_{x,y}}_{=0}$$

for independent x, y

$$\sigma_f^2 = \left(\frac{df}{dx}\right)^2\sigma_x^2 + \left(\frac{df}{dy}\right)^2\sigma_y^2$$

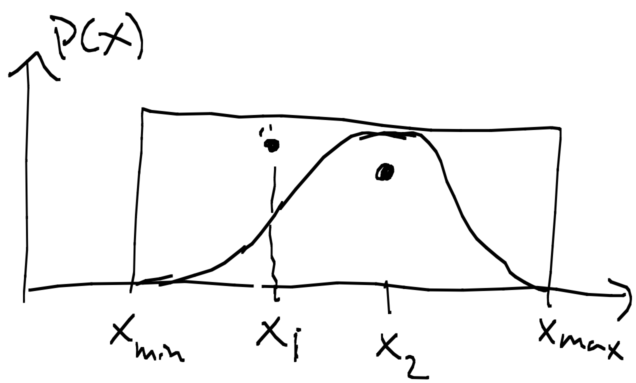
$$\sigma_f = \sqrt{\left(\frac{df}{dx}\right)^2\sigma_x^2 + \left(\frac{df}{dy}\right)^2\sigma_y^2}$$

$$\longrightarrow \sigma_f = \sqrt{\sum_i \left(\frac{df}{dx_i}\right)^2 \sigma_{x_i}^2}$$ for arbitrary number of variables.

Modeling sampling

given a PDF $P(x)$, how should we randomly sample values such that the histogram of x_i's $\longrightarrow P(x)$ for large N?

Von Neumann sampling:



1. pick random value of $x = x'$, $x_{min} < x' < x_{max}$
2. Pick random value y between 0 and $\max(P(x))$ within the range $x_{min} < x < x_{max}$
3. If $y < P(x')$ then x' is a valid random sample of $P(x)$. Otherwise, repeat.

Reflection and Refraction

Wave equation in matter:

$$\mu = \mu_r \mu_0 \qquad \frac{\partial^2 E}{\partial z^2} = \mu_r \mu_0 \varepsilon_r \varepsilon_0 \frac{\partial^2 E}{\partial t^2}$$
$$\varepsilon = \varepsilon_r \varepsilon_0$$

$$\text{Velocity} = \frac{1}{\sqrt{\mu_r \varepsilon_r}\sqrt{\mu_0 \varepsilon_0}} = \frac{1}{\sqrt{\mu_r \varepsilon_r}} c \equiv \frac{c}{n}$$

plane wave: $E = E_0 e^{i(kz - \omega t)} = E_0 e^{ik(z - \frac{\omega}{k} t)}$

$$\text{Velocity} = \frac{\omega}{k} = \frac{c}{n}$$

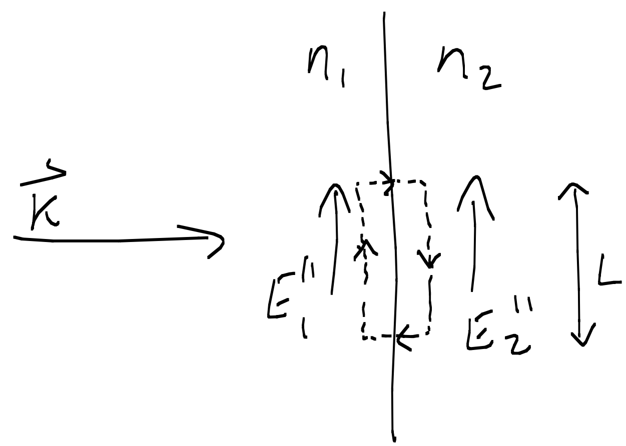
$\omega \to \frac{\omega_0}{n}$?

$k \to k_0 n$?

$\omega_0, k_0 \equiv$ free space values

Boundary Conditions on $\vec{E}$ at interfaces

Faraday's Law:



$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}$$

$$E_1^{\parallel} L - E_2^{\parallel} L \to 0$$ for path w/ negligible area

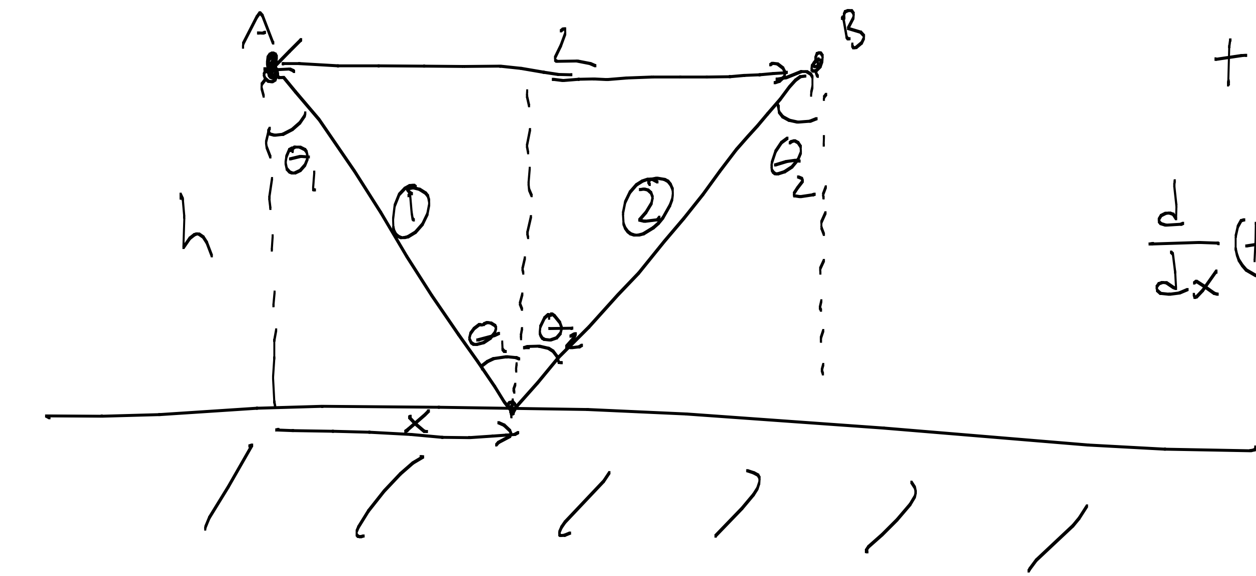
$$E_1^{\parallel} = E_2^{\parallel}$$

So $\vec{E}$ must have same freq. of oscillation on both sides

$$\Longrightarrow \omega_1 = \omega_2$$

So $K \to K_0 n$

Oblique reflection



$$\text{time} = \frac{n\sqrt{h^2+x^2}}{c} + \frac{n\sqrt{h^2+(L-x)^2}}{c}$$

$$\frac{d}{dx}(\text{time}) = \frac{2n\,x}{2c\sqrt{h^2+x^2}} - \frac{2n(L-x)}{2c\sqrt{h^2+(L-x)^2}} = 0$$

$$\frac{x}{\sqrt{h^2+x^2}} = \frac{(L-x)}{\sqrt{h^2+(L-x)^2}}$$

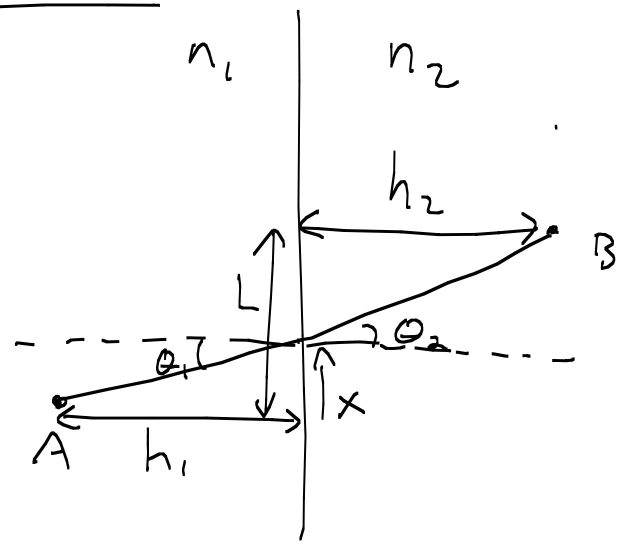
$$\sin\theta_1 = \sin\theta_2$$

$$\theta_1 = \theta_2$$

Fermat's Principle:

Light follows paths through space a transit time that is a local extrema (minimum or maximum!)

Refraction



$$\text{time} = \frac{n_1\sqrt{h_1^2+x^2}}{c} + \frac{n_2\sqrt{h_2^2+(L-x)^2}}{c}$$

"Stationarize" time

$$n_1 \frac{x}{\sqrt{h_1^2+x^2}} = n_2 \frac{L-x}{\sqrt{h_2^2+(L-x)^2}}$$

"Snell's Law" $n_1 \sin\theta_1 = n_2 \sin\theta_2$

Note: for $n_1 > n_2$

$$\theta_2 = \sin^{-1}\left(\frac{n_1}{n_2}\sin\theta_1\right)$$

If $\sin\theta_1 > \frac{n_2}{n_1}$, θ_2 is undefined!

"total internal reflection"