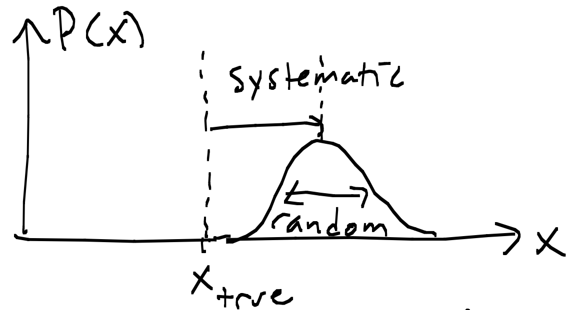
Measurement Error

systematic error $\longrightarrow$ accuracy

random error $\longrightarrow$ precision



$P(x)$: Probability distribution function (PDF)

$$\int_{-\infty}^{\infty} P(x)\, dx = 1$$

"moments"

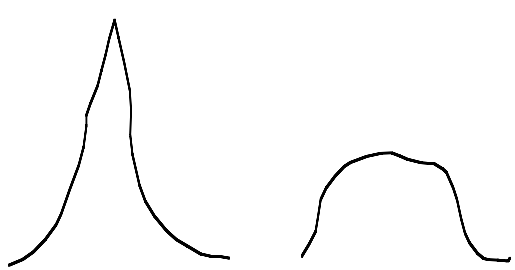
1^{st}: $$\frac{\int_{-\infty}^{\infty} P(x)\, x\, dx}{\int_{-\infty}^{\infty} P(x)\, dx} = \bar{x}$$ "mean"

2nd: $\dfrac{\int_{-\infty}^{\infty} P(x)(x-\bar{x})^2 dx}{\int_{-\infty}^{\infty} P(x) dx} = \sigma^2$ "variance"

σ "standard deviation"

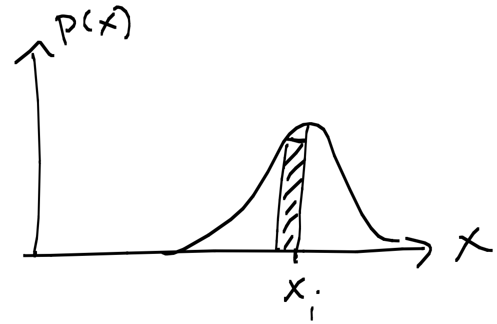
3rd: $\dfrac{\int_{-\infty}^{\infty} P(x)(x-\bar{x})^3 dx}{\sigma^3 \int_{-\infty}^{\infty} P(x) dx}$ "skew" (asymmetry)

4th: $\dfrac{\int_{-\infty}^{\infty} P(x)(x-\bar{x})^4 dx}{\sigma^4 \int_{-\infty}^{\infty} P(x) dx} - 3$ "Kurtosis" (peaked-ness)



<u>Sampling</u>

Mean: $$\frac{\int_{-\infty}^{\infty} x \, P(x)\,dx}{\int_{-\infty}^{\infty} P(x)\,dx}$$



$$\xrightarrow{\text{Sampling}} \frac{\sum_i^N x_i}{\sum_i^N 1} = \frac{\sum_i^N x_i}{N}$$

Variance: $$\frac{\int_{-\infty}^{\infty} (x-\bar{x})^2 P(x)\,dx}{\int_{-\infty}^{\infty} P(x)\,dx} \Longrightarrow \frac{\sum_i^N (x_i-\bar{x})^2}{N-1}$$

"mode": most common value

"median": middle value

Combining Measurements

$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\bar{x})^2}{2\sigma^2}}$$

$$P(x_A) = \frac{1}{\sigma_A\sqrt{2\pi}} e^{-\frac{(x_A-\bar{x})^2}{2\sigma_A^2}} \qquad P(x_B) = \frac{1}{\sigma_B\sqrt{2\pi}} e^{-\frac{(x_B-\bar{x})^2}{2\sigma_B^2}}$$

$$P(x_A, x_B) = P(x_A)P(x_B) \propto e^{-\frac{1}{2}\left[\frac{(x_A-\bar{x})^2}{2\sigma_A^2} + \frac{(x_B-\bar{x})^2}{2\sigma_B^2}\right]}$$

$$\frac{d}{d\bar{x}}\left[\frac{(x_A-\bar{x})^2}{\sigma_A^2} + \frac{(x_B-\bar{x})^2}{\sigma_B^2}\right] = 0$$

$$\frac{-2(x_A-\bar{x})}{\sigma_A^2} + \frac{-2(x_B-\bar{x})}{\sigma_B^2} = 0$$

$$w_A \equiv \frac{1}{\sigma_A^2}, \quad w_B \equiv \frac{1}{\sigma_B^2}$$

$$w_A(x_A - \bar{x}) + w_B(x_B - \bar{x}) = 0$$

$$w_A x_A + w_B x_B - \bar{x}(w_A + w_B) = 0$$

$$\bar{x} = \frac{w_A x_A + w_B x_B}{w_A + w_B}$$

$$\longrightarrow \quad \bar{x} = \frac{\sum_i w_i x_i}{\sum_i w_i} \quad \text{where} \quad w_i \equiv \frac{1}{\sigma_i^2}$$

Standard deviation of the mean

$$P(x_1, x_2, \ldots x_N) = \prod_i^N P_i() = e^{-\frac{(x-\bar{x})^2}{2\sigma_1^2}} e^{-\frac{(x-\bar{x})^2}{2\sigma_2^2}} \cdots e^{-\frac{(x-\bar{x})^2}{2\sigma_N^2}}$$

$$= e^{-\left[\sum_i \frac{1}{\sigma_i^2}\right]\frac{(x-\bar{x})^2}{2}}$$

$$\longrightarrow \quad \frac{1}{\sigma_{\bar{x}}^2} = \sum_i^N \frac{1}{\sigma_i^2}$$

If all σ_i's are same, $\quad \frac{1}{\sigma_{\bar{x}}^2} = \frac{N}{\sigma^2}$

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{N}}$$

Propagation of Errors

$f(x, y)$

$$\sigma_f^2 = \frac{1}{N} \sum_i^N \left(f(x_i, y_i) - f(\bar{x}, \bar{y})\right)^2$$

$$f(x_i, y_i) \sim f(\bar{x}, \bar{y}) + \frac{df}{dx}(x_i - \bar{x}) + \frac{df}{dy}(y_i - \bar{y})$$

$$\sigma_f^2 = \frac{1}{N} \sum_i^N \left[\cancel{f(\bar{x}, \bar{y})} + \frac{df}{dx}(x_i - \bar{x}) + \frac{df}{dy}(y_i - \bar{y}) - \cancel{f(\bar{x}, \bar{y})}\right]^2$$

$$= \frac{1}{N} \sum_i^N \left[\left(\frac{df}{dx}\right)^2 (x_i - \bar{x})^2 + \left(\frac{df}{dy}\right)^2 (y_i - \bar{y})^2 + 2 \frac{df}{dx} \frac{df}{dy} (x_i - \bar{x})(y_i - \bar{y})\right]$$

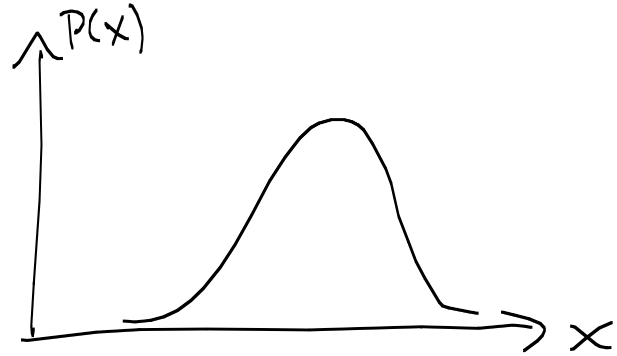
$$= \left(\frac{df}{dx}\right)^2 \frac{1}{N} \sum_i^N (x_i - \bar{x})^2 + \left(\frac{df}{dy}\right)^2 \frac{1}{N} \sum_i^N (y_i - \bar{y})^2 + 2 \frac{df}{dx} \frac{df}{dy} \frac{1}{N} \sum_i^N (x_i - \bar{x})(y_i - \bar{y})$$

$$\sigma_f^2 = \left(\frac{df}{dx}\right)^2 \sigma_x^2 + \left(\frac{df}{dy}\right)^2 \sigma_y^2 + 0$$ (if x and y are independent)

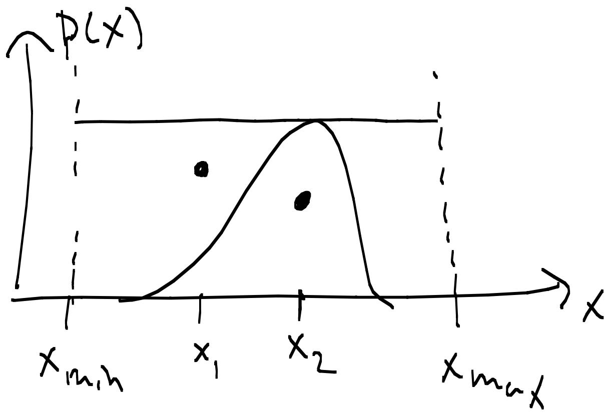
$$\sigma_f = \sqrt{\left(\frac{df}{dx}\right)^2 \sigma_x^2 + \left(\frac{df}{dy}\right)^2 \sigma_y^2}$$

$$\longrightarrow \sigma_f = \sqrt{\sum_i \left(\frac{df}{dx_i}\right)^2 \sigma_{x_i}^2}$$

Modeling Sampling



Von Neumann sampling



Wave equation

$\mu = \mu_r \mu_0$

$\varepsilon = \varepsilon_r \cdot \varepsilon_0$

$$\frac{\partial^2 E}{\partial z^2} = \mu_r \varepsilon_r \mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2}$$

$$\text{Velocity} = \frac{1}{\sqrt{\mu_r \varepsilon_r}\sqrt{\mu_0 \varepsilon_0}} = \frac{c}{\sqrt{\mu_r \varepsilon_r}} = \frac{c}{n}$$

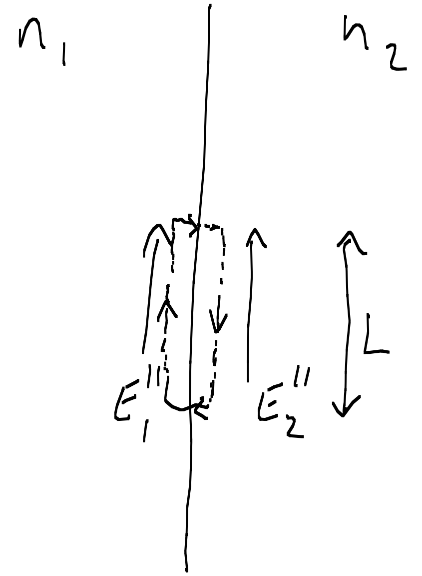
$n \rightarrow$ "index of refraction"

Plane wave $\quad E = E_0 e^{i(kz - \omega t)} = E_0 e^{ik(z - \frac{\omega}{k} t)}$

$$\text{Velocity} = \frac{\omega}{k} = \frac{c}{n}$$

$\omega \rightarrow \frac{\omega_0}{n}$? $\qquad k \rightarrow k_0 n$?

Boundary Conditions on $\vec{E}$ at interfaces



Faraday's Law

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \Phi_B$$

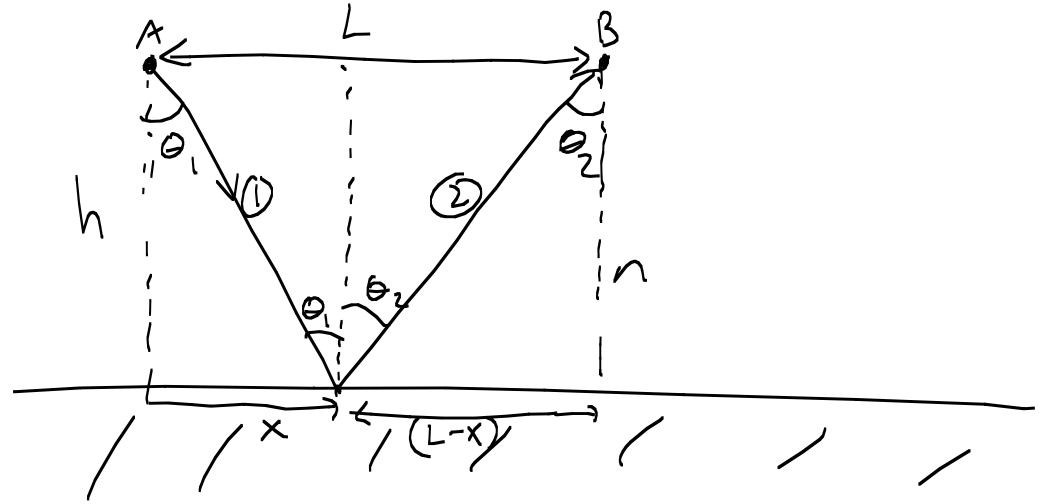
$$E_1^{\parallel} L - E_2^{\parallel} L \longrightarrow 0$$ for a path w/ negligible area

$$E_1^{\parallel} = E_2^{\parallel}$$

$$\Longrightarrow \quad \omega_1 = \omega_2$$

so $K \longrightarrow K_0 n$

Oblique reflection



$$\text{time} = \frac{\sqrt{h^2+x^2}}{c/n} + \frac{\sqrt{h^2+(L-x)^2}}{c/n}$$

$$\frac{d}{dx}(\text{time}) = \frac{n}{c}\frac{1}{2}\frac{2x}{\sqrt{h^2+x^2}} - \frac{n}{c}\frac{1}{2}\frac{2(L-x)}{\sqrt{h^2+(L-x)^2}}$$

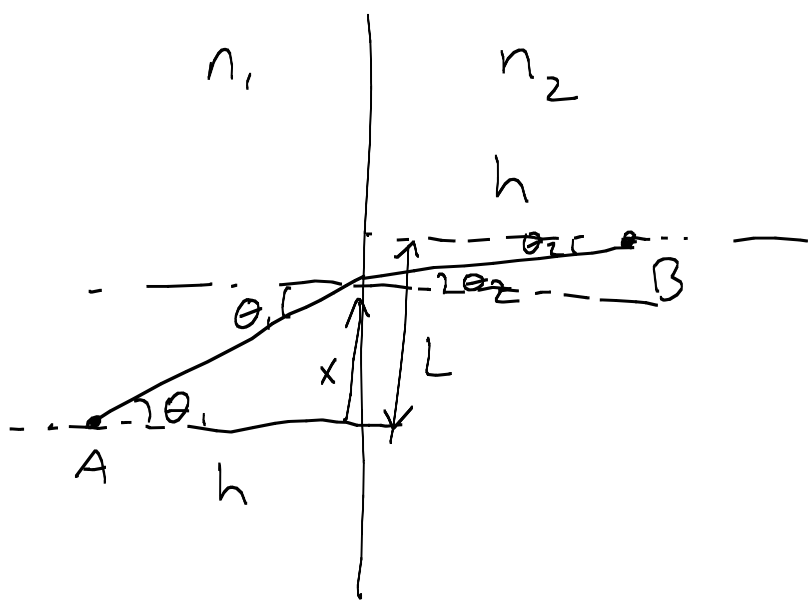
$$= 0$$

$$\frac{x}{\sqrt{h^2+x^2}} = \frac{L-x}{\sqrt{h^2+(L-x)^2}}$$

$$\sin\theta_1 = \sin\theta_2$$

$$\theta_1 = \theta_2$$

Refraction



$$\text{time} = \frac{\sqrt{h^2+x^2}}{c/n_1} + \frac{\sqrt{h^2+(L-x)^2}}{c/n_2}$$

$$\frac{d}{dx}(\text{time}) = 0$$

$$n_1 \frac{x}{\sqrt{h^2+x^2}} = n_2 \frac{L-x}{\sqrt{h^2+(L-x)^2}}$$

"Snell's Law" $\quad n_1 \sin\theta_1 = n_2 \sin\theta_2$

Note: $\quad \theta_2 = \sin^{-1}\left(\frac{n_1}{n_2}\sin\theta_1\right)$

If $\sin\theta_1 > \frac{n_2}{n_1}$, θ_2 is undefined!

"total internal reflection"