Integral Form of Maxwell's Equations

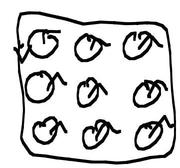
$$\oiint \vec{E}\cdot d\vec{S} = \frac{Q}{\varepsilon_0}$$ "Gauss' Law"

$$\oint \vec{E}\cdot d\vec{l} = -\frac{d\Phi_B}{dt}$$ "Faraday's Law"

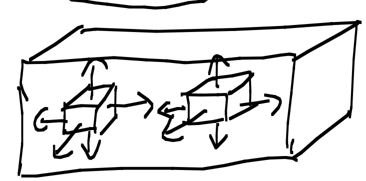
$$\oiint \vec{B}\cdot d\vec{S} = 0$$

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I + \mu_0\varepsilon_0 \frac{d\Phi_E}{dt}$$ "Ampere's Law"

"Stokes' Thm" $$\oint \vec{A}\cdot d\vec{l} = \iint \vec{\nabla}\times\vec{A}\cdot d\vec{S}$$



"divergence Thm" $$\oiint \vec{A}\cdot d\vec{S} = \iiint \vec{\nabla}\cdot\vec{A}\, dV$$



Maxwell's Eqns in Differential form

$$\oiint \vec{E}\cdot d\vec{S} = \iiint \vec{\nabla}\cdot\vec{E}\, dV = \frac{Q}{\varepsilon_0} = \frac{\iiint \rho\, dV}{\varepsilon_0} \Rightarrow \boxed{\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}}$$

$$\oint \vec{E}\cdot d\vec{l} = \iint \vec{\nabla}\times\vec{E}\cdot d\vec{S} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}\iint \vec{B}\cdot d\vec{S} \Rightarrow \boxed{\vec{\nabla}\times\vec{E} = -\frac{d\vec{B}}{dt}}$$

$$\oiint \vec{B}\cdot d\vec{S} = \iiint \vec{\nabla}\cdot\vec{B}\, dV = 0 \Rightarrow \boxed{\vec{\nabla}\cdot\vec{B} = 0}$$

$$\oint \vec{B}\cdot d\vec{l} = \iint \vec{\nabla}\times\vec{B}\cdot d\vec{S} = \mu_0 I + \mu_0\varepsilon_0 \frac{d\Phi_E}{dt} = \iint \left[\mu_0\vec{J} + \mu_0\varepsilon_0\frac{d\vec{E}}{dt}\right]\cdot d\vec{S}$$

$$\Rightarrow \boxed{\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{d\vec{E}}{dt}}$$

Wave Equation

$\vec{\nabla}\times$ (Faraday's Law):

in free space

$$\vec{\nabla}\times\vec{\nabla}\times\vec{E} = \vec{\nabla}\times\left(-\frac{d\vec{B}}{dt}\right) = -\frac{d}{dt}\left(\vec{\nabla}\times\vec{B}\right) = -\frac{d}{dt}\left(\cancelto{0}{\mu_0\vec{J}} + \mu_0\varepsilon_0\frac{d\vec{E}}{dt}\right)$$

in free space

$$\vec{\nabla}\left(\cancelto{0}{\vec{\nabla}\cdot\vec{E}}\right) - \nabla^2\vec{E} = -\mu_0\varepsilon_0\frac{d^2\vec{E}}{dt^2}$$

$$\nabla^2\vec{E} = \mu_0\varepsilon_0\frac{d^2\vec{E}}{dt^2}$$

$$\mu_0\varepsilon_0 \equiv \frac{1}{c^2} \qquad \left[\mu_0\varepsilon_0\right] = \frac{1}{\text{Velocity}^2}$$

$$\nabla^2\vec{E} = \frac{1}{c^2}\frac{\partial^2\vec{E}}{dt^2} \qquad \text{"Wave equation"}$$

1-D $\left(\nabla^2 \to \frac{\partial^2}{\partial z^2}\right)$ solution: $F(z \pm ct)$ "D'Alembert" solution

"Simple" example in 1-D:

$$E = E_0 e^{i(kz \pm \omega t)} = E_0 e^{ik(z \pm \frac{\omega}{k} t)} \qquad \text{"plane wave"}$$

$$\frac{\omega}{k} = c \qquad k \rightarrow \text{"wavenumber"} \qquad k = \frac{2\pi}{\lambda}$$

A more general solution which allows a confined "beam":

$$\vec{E}(x,y,z,t) = A(x,y,z)\, e^{i\omega t}$$

$$\nabla^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2}$$

$$\nabla^2 A \cdot \cancel{e^{i\omega t}} = -\frac{\omega^2}{c^2} A \cancel{e^{i\omega t}}$$

$$\nabla^2 A + k^2 A = 0 \qquad \text{"Helmholtz eqn"}$$

$$\left(\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right)$$

Assume $\quad A = u(x,y,z)\,e^{ikz}$

$$\nabla_T^2 u e^{ikz} + \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z} e^{ikz} + iku e^{ikz}\right) + k^2 u e^{ikz} = 0$$

x,y

$$\nabla_T^2 u \cancel{e^{ikz}} + \frac{\partial^2 u}{\partial z^2}\cancel{e^{ikz}} + ik\frac{\partial u}{\partial z}\cancel{e^{ikz}} + ik\frac{\partial u}{\partial z}\cancel{e^{ikz}} \cancel{- k^2 u e^{ikz}} \cancel{+ k^2 u e^{ikz}} = 0$$

$$\nabla_T^2 u + \cancel{\frac{\partial^2 u}{\partial z^2}} + 2ik\frac{\partial u}{\partial z} = 0$$

neglect for "slowly varying" solution

$$r \equiv \sqrt{x^2+y^2}: \qquad \frac{\partial^2 u}{\partial r^2} + i2k\frac{\partial u}{\partial z} = 0$$

$$\frac{\partial^2 u}{\partial r^2} = -i\,2k\frac{du}{\partial z}$$ looks familiar!

Note:

If $t \equiv z$ $\quad -\frac{\hbar^2}{2m} \equiv 1$ $\quad \hbar \equiv 2k$ $\Rightarrow$ $-\frac{\hbar^2}{2m}\frac{\partial^2 u}{\partial r^2} = \frac{\hbar}{i}\frac{\partial u}{\partial t}$

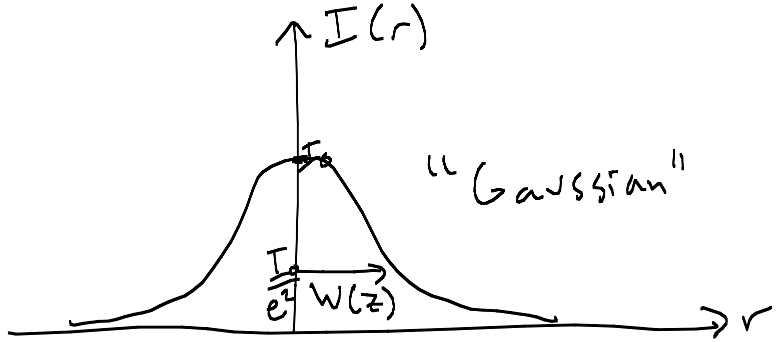
equivalent to Schrödinger's Eqn for free partizle

Solution:

$$u = E_0 e^{i(\text{phase})} e^{-\frac{r^2}{W^2(z)}}$$

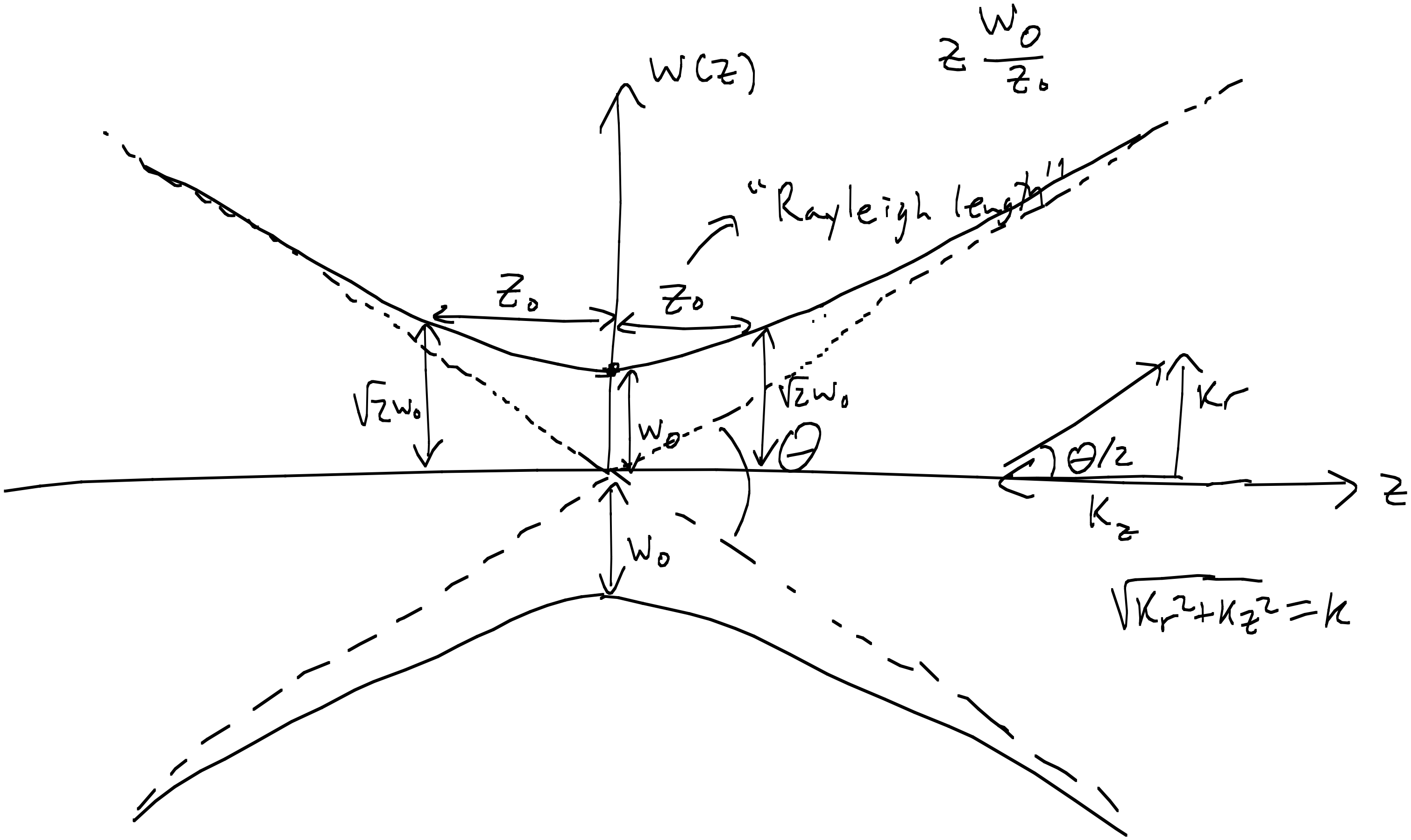
$$\left(E = u e^{i(kz-wt)}\right)$$

We measure $I = |\vec{E}|^2 = I_0 e^{-\frac{2r^2}{W^2(z)}}$



"Gaussian"

$$W(z) = W_0 \sqrt{1 + \left(\frac{z}{z_0}\right)^2}$$



Uncertainty Relation

$$\Delta r \, \Delta k_r \gtrsim \text{"}1\text{"}$$

$$I = I_0 e^{-\frac{2r^2}{w_0}}$$

$$\Delta r = w_0$$

Δk_r given by Fourier transform

$$\propto e^{-\frac{k^2}{8/w_0^2}}$$

$$\Delta k_r = \frac{4}{w_0} = 2k_r = 2k \sin\frac{\theta}{2} \sim k\theta$$

$$\Rightarrow \Delta r \Delta k_r = 4$$

$$w_0 = \frac{4}{k\theta} = \frac{4\lambda}{2\pi\theta} = \frac{2\lambda}{\pi\theta}$$

Rayleigh length

$$k_z = k\cos\frac{\theta}{2} \sim k\left(1 - \frac{(\theta/2)^2}{2}\right) = k\left(1 - \frac{\theta^2}{8}\right)$$

$$\Delta k_z = k\frac{\theta^2}{8}$$

$\Delta k_z \, \Delta z = 2$ by redefining uncertainty so that $w(z) = \sqrt{2}\, w_0$

$$\Delta z = 2 z_0 = \frac{2}{\Delta k_z} = \frac{2}{k\frac{\theta^2}{8}} = \frac{16}{k\theta^2}$$

$$z_0 = \frac{8}{k\theta^2} = \frac{8\lambda}{2\pi\theta^2} = \frac{4\lambda}{\pi\theta^2}$$

$$z_0 = \frac{4\lambda}{\pi\left(\frac{2\lambda}{\pi w_0}\right)^2} = \frac{4\lambda\pi^2 w_0^2}{\pi 4\lambda^2} = \frac{\pi w_0^2}{\lambda}$$

Lab 0