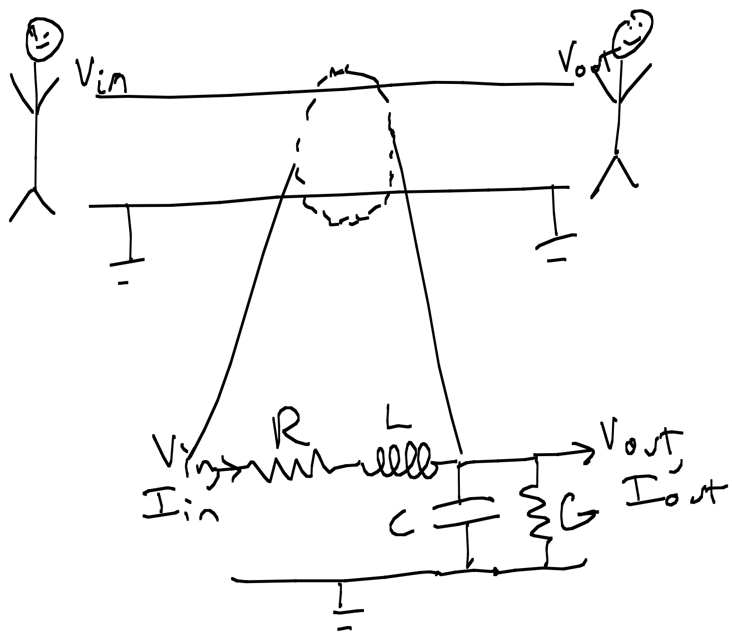
Telegrapher's Equation



$$\frac{\Delta V}{\Delta x} \rightarrow \frac{\partial V}{\partial x} = -IR - L\frac{\partial I}{\partial t} \quad (1)$$

$$\frac{\Delta I}{\Delta x} \rightarrow \frac{\partial I}{\partial x} = -GV - C\frac{\partial V}{\partial t} \quad (2)$$

$$\frac{\partial}{\partial x}(1): \quad \frac{\partial^2 V}{\partial x^2} = -R\frac{\partial I}{\partial x} - L\frac{\partial^2 I}{\partial x \partial t}$$

$$\frac{\partial}{\partial t}(2): \quad \frac{\partial^2 I}{\partial x \partial t} = -G\frac{\partial V}{\partial t} - C\frac{\partial^2 V}{\partial t^2}$$

$$\frac{\partial^2 V}{\partial x^2} = -R\left(-GV - C\frac{\partial V}{\partial t}\right) - L\left(-G\frac{\partial V}{\partial t} - C\frac{\partial^2 V}{\partial t^2}\right)$$

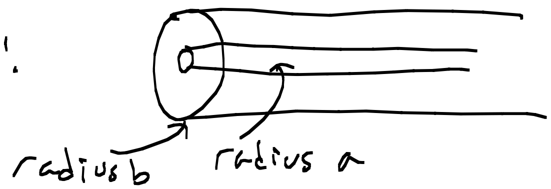
$$\frac{\partial^2 V}{\partial x^2} = LC\frac{\partial^2 V}{\partial t^2} + (LG + CR)\frac{\partial V}{\partial t} + GRV$$

"telegrapher's equation"

Simplest Case: $R = G = 0$

$$\frac{\partial^2 V}{\partial x^2} = LC \frac{\partial^2 V}{\partial t^2}$$ wave eqn! $\frac{1}{s^2} = LC$ Sol'ns: $V(x,t) = V(x \pm st)$ "d'Alembert"

what is s?!



coaxial cable

Capacitance: $C = \frac{q}{V}$

Gauss' law: $\oint \vec{E} \cdot d\vec{s} = \frac{q}{\varepsilon}$

$E\, 2\pi r = \frac{q}{\varepsilon} \rightarrow E = \frac{q}{2\pi\varepsilon r}$

$$V = -\int_a^b \vec{E} \cdot d\vec{r} = \int_a^b \frac{q}{2\pi\varepsilon r} dr$$

$$V = \frac{q}{2\pi\varepsilon} \ln\frac{b}{a}$$

$$C = \frac{q}{V} = \frac{2\pi\varepsilon}{\ln b/a}$$

inductance: $L = \frac{\Phi_B}{I}$

Ampere's Law $\oint \vec{B} \cdot d\vec{l} = \mu I$

$B\, 2\pi r = \mu I \rightarrow B = \frac{\mu I}{2\pi r}$

$$\Phi_B = \int \vec{B} \cdot d\vec{s} = \int_a^b \frac{\mu I}{2\pi r} dr$$

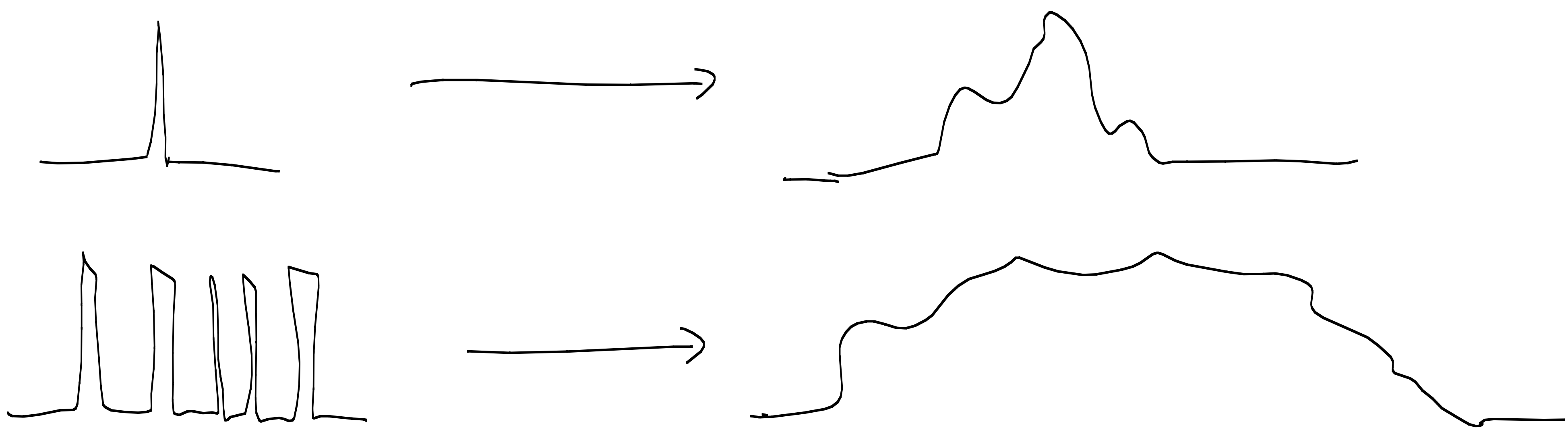
$$= \frac{\mu I}{2\pi} \ln b/a$$

$$L = \frac{\Phi_B}{I} = \frac{\mu}{2\pi} \ln b/a$$

$$LC = \frac{1}{s^2} = \frac{\cancel{2\pi}\varepsilon}{\cancel{\ln b/a}} \frac{\mu}{\cancel{2\pi}} \cancel{\ln b/a} = \varepsilon\mu = \frac{1}{c^2/n^2} \qquad s = \frac{c}{n}$$

In general, $R, G \neq 0$

nonzero terms cause attenuation and distortion



can we choose R, L, C, G such that distortion is eliminated?

Ansatz

$$\frac{\partial^2 V}{\partial x^2} = LC\frac{\partial^2 V}{\partial t^2} + (LG+CR)\frac{\partial V}{\partial t} + GRV$$

"Telegrapher's equation"

$$V(x,t) = u(x,t)\, e^{-\left(\frac{LG+CR}{2LC}\right)t}$$

$$\frac{\partial^2 u}{\partial x^2} e^{-(\ )t} = LC\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t} e^{-(\ )t} - \frac{LG+CR}{2LC} u e^{-(\ )t}\right)$$
$$+ (LG+CR)\left(\frac{\partial u}{\partial t} e^{-(\ )t} - \frac{LG+CR}{2LC} u e^{-(\ )t}\right) + GRue^{-(\ )t}$$

$$\frac{\partial^2 u}{\partial x^2} \cancel{e^{-(\ )t}} = LC\left[\frac{\partial^2 u}{\partial t^2} \cancel{e^{-(\ )t}} - \frac{LG+CR}{2LC}\frac{\partial u}{\partial t} \cancel{e^{-(\ )t}} - \frac{LG+CR}{2LC}\left(\frac{\partial u}{\partial t} \cancel{e^{-(\ )t}} - \frac{LG+CR}{2LC} \cancel{ue^{-(\ )t}}\right)\right]$$
$$+ (LG+CR)\left(\frac{\partial u}{\partial t} \cancel{e^{-(\ )t}} - \frac{LG+CR}{2LC} u \cancel{e^{-(\ )t}}\right) + GRu\cancel{e^{-(\ )t}}$$

$$\frac{\partial^2 u}{\partial x^2} = LC\frac{\partial^2 u}{\partial t^2} + \left(GR - \frac{(LG+CR)^2}{4LC}\right)u$$

$$\frac{\partial^2 u}{\partial x^2} = LC\frac{\partial^2 u}{\partial t^2} + \left(\frac{4GRLC - (L^2G^2 + 2LGCR + C^2R^2)}{4LC}\right)u$$

$$\underbrace{\frac{\partial^2 u}{\partial x^2} = LC\frac{\partial^2 u}{\partial t^2}}_{\text{wave eqn.}} - \left(\frac{(LG - CR)^2}{4LC}\right)u$$

$$V(x,t) = u(x,t)\,e^{-\left(\frac{LG+CR}{2LC}\right)t}$$

If $LG = CR$, $u(x,t)$ is soln to wave equation

→ no distortion! (Heaviside 1893)

Submarine cables must be inductively loaded to increase L to satisfy Heaviside's requirement.