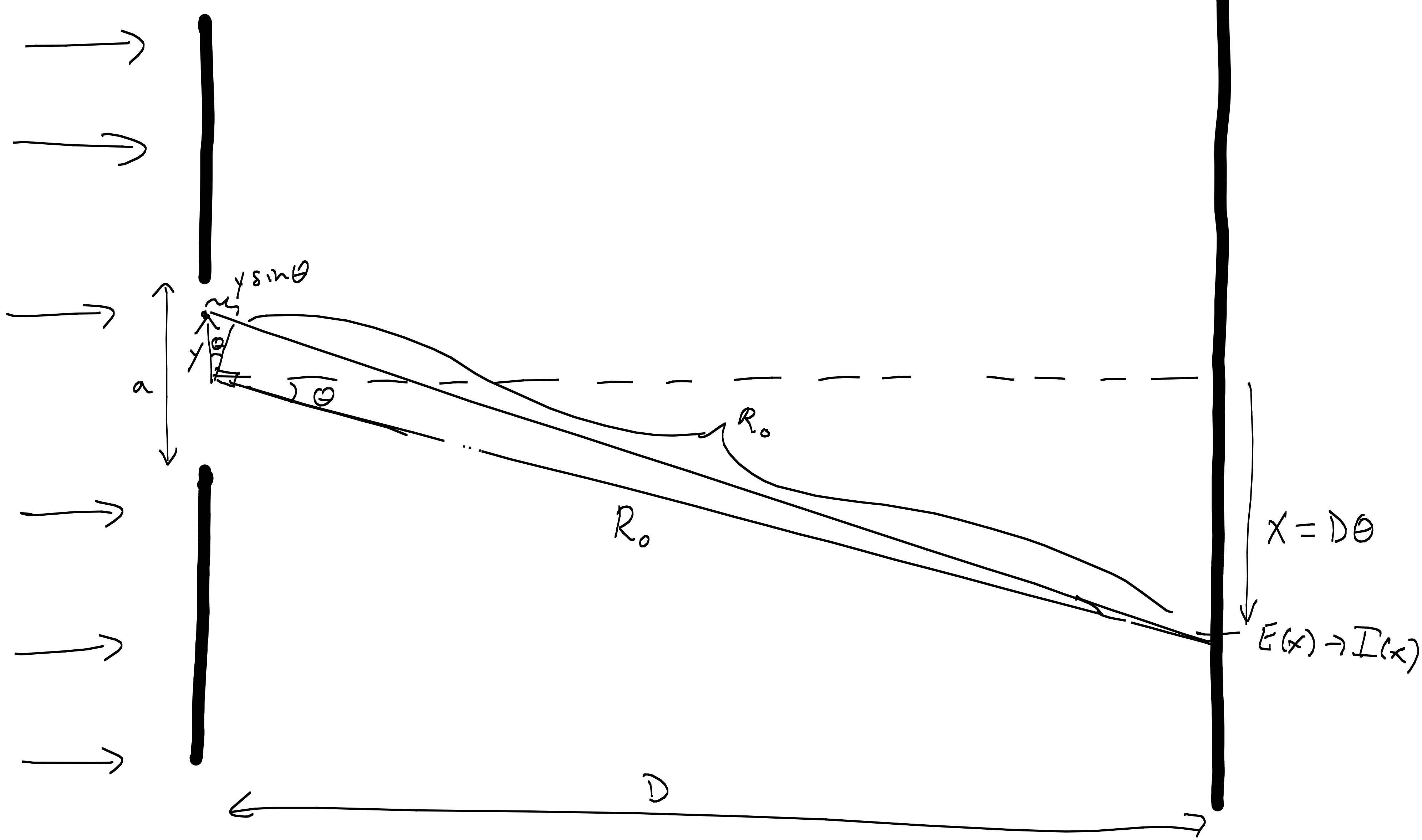
Self-Interference: Diffraction



E-field at $x = D\theta$

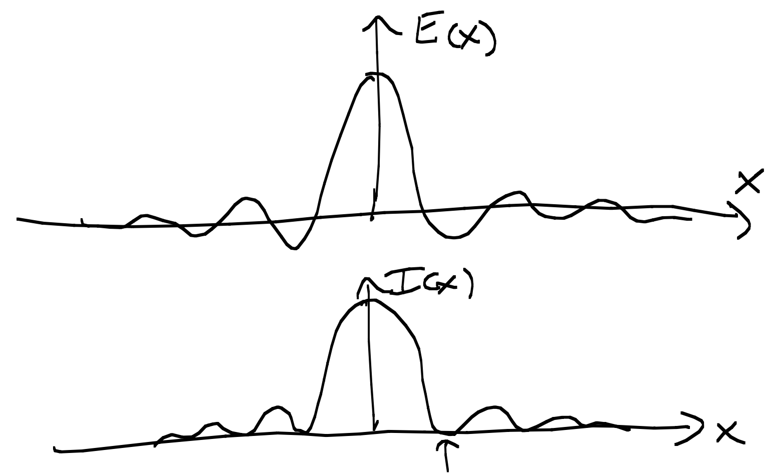
$$E \propto \int_{-a/2}^{+a/2} e^{ik(R_0 + y\sin\theta)}\,dy = e^{ikR_0}\int_{-a/2}^{+a/2} e^{iky\sin\theta}\,dy = e^{ikR_0}\left.\frac{e^{iky\sin\theta}}{ik\sin\theta}\right|_{-a/2}^{+a/2}$$

$$= e^{ikR_0}\frac{a}{2}\,\frac{e^{\frac{ik\sin\theta a}{2}} - e^{\frac{-ik\sin\theta a}{2}}}{ik\sin\theta\frac{a}{2}} \qquad \alpha \equiv \frac{k\sin\theta a}{2} \sim \frac{k\theta a}{2}$$

$$= e^{ikR_0}\frac{a}{\cancel{2}}\,\frac{\cancel{2}\sin\alpha}{\alpha}$$

$$\boxed{\frac{e^{ix} - e^{-ix}}{i} = 2\sin x}$$

$$= a e^{ikR_0}\underbrace{\frac{\sin\alpha}{\alpha}}_{\text{"sinc"}}$$

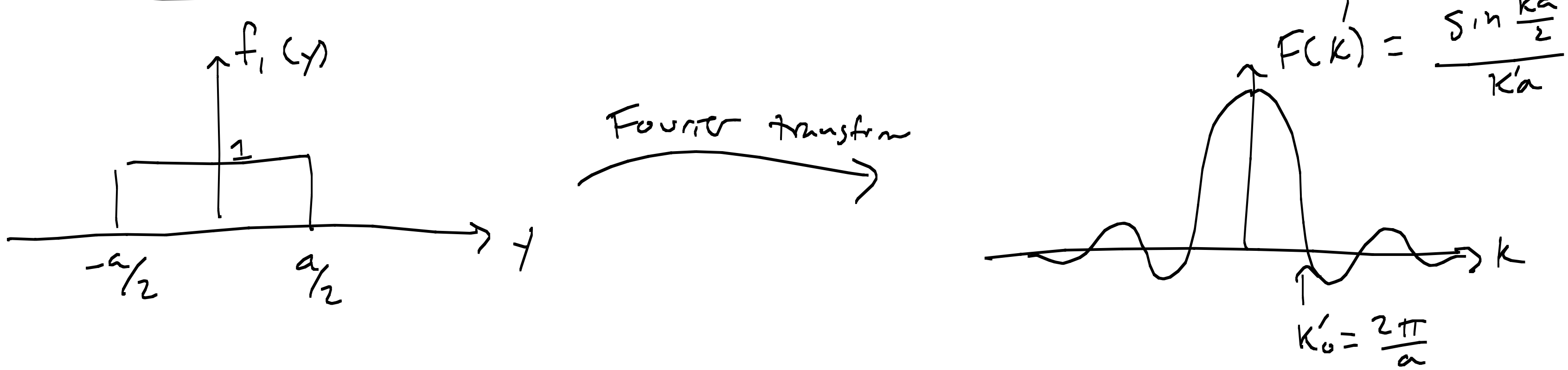


$$\frac{ka\theta_0}{2} = \pi$$

$$\theta_0 = \frac{2\pi}{ka} = \frac{\lambda}{a}$$

$$x_0 = D\lambda/a$$

Fourier Analysis

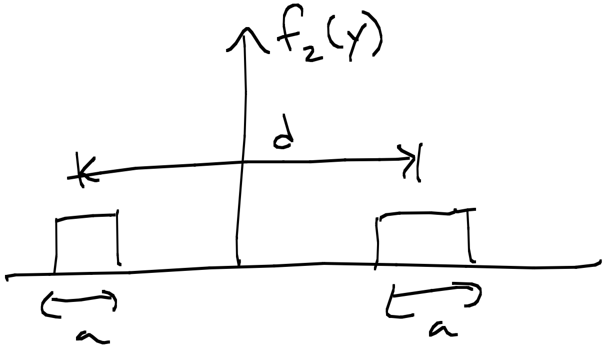


to calculate diffraction pattern, take F.T. of aperture transmission fn. Then $k' \to \frac{2\pi x}{D\lambda}$

example: (above)

$$F(k') = \frac{\sin \frac{k'a}{2}}{k'a} \qquad k' \to \frac{2\pi x}{D\lambda} \qquad \frac{\sin \frac{2\pi x}{D\lambda}\frac{a}{2}}{\frac{2\pi x}{D\lambda} a} = \frac{\sin \frac{\pi a}{\lambda}\theta}{2\pi \frac{a\theta}{\lambda}} = \frac{\sin \frac{ka\theta}{2}}{ka\theta}$$

Example: Young's two-slit expt



$$F.T.\left[f_2(y)\right] = F.T.\left[f_1\left(y+\frac{d}{2}\right) + f_1\left(y-\frac{d}{2}\right)\right]$$

$$= e^{-\frac{ik'd}{2}} F.T.\left[f_1(y)\right] + e^{+\frac{ik'd}{2}} F.T.\left[f_1(y)\right]$$

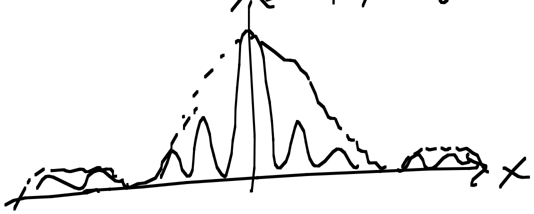
$$= F.T.\left[f_1(y)\right]\left(e^{\frac{ik'd}{2}} + e^{-\frac{ik'd}{2}}\right)$$

$$= 2 \cdot F.T.\left[f_1(y)\right] \cos\frac{k'd}{2}$$

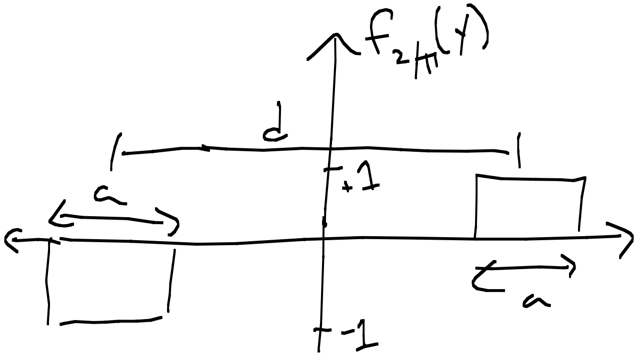
$$k' \to \frac{2\pi x}{D\lambda} \qquad = 2 \cdot F.T.\left[f_1(y)\right] \cos\frac{2\pi x}{D\lambda}\frac{d}{2}$$

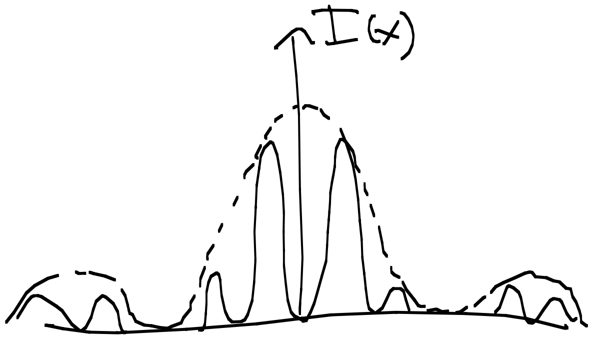
$$E(x) = \text{`` \quad ''} \quad \cos\frac{\overset{\frac{2\pi}{\lambda}}{k}x}{D}\frac{d}{2}$$

$I(x) \propto \text{sinc}^2\alpha \cdot \cos\frac{kxd}{D}$



Example: $\Delta\phi = \pi$





what is diffracted image?

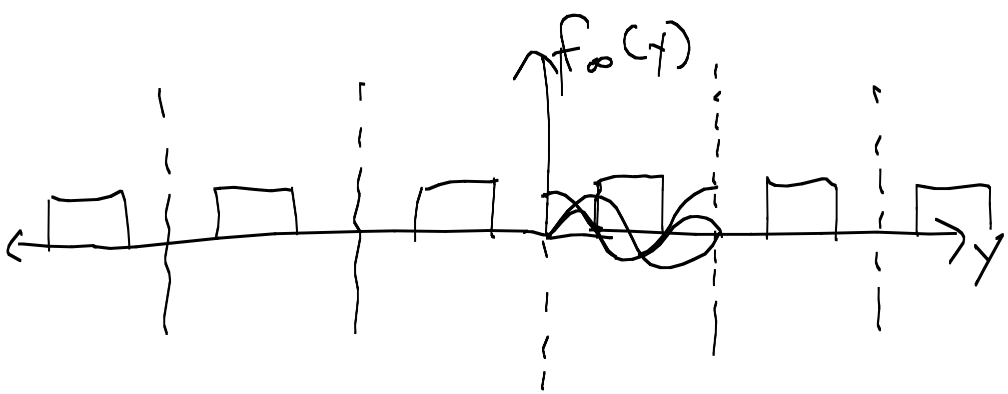
$$F.T.\left[f_1\left(y-\frac{d}{2}\right) - f_1\left(y+\frac{d}{2}\right)\right]$$

$$= e^{ik'\frac{d}{2}}\, F.T.\left[f_1(y)\right] - e^{-ik'\frac{d}{2}}\, F.T.\left[f_1(y)\right]$$

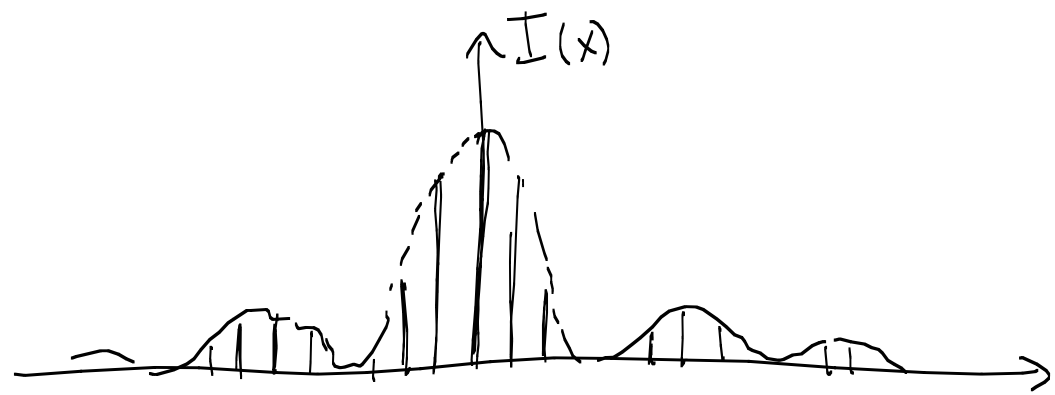
$$= F.T.\left[f_1(y)\right]\left(e^{ik'\frac{d}{2}} - e^{-ik'\frac{d}{2}}\right)$$

$$= 2i\, \underbrace{\sin\frac{kd}{2}}_{\substack{\text{interference} \\ \text{between 2} \\ \text{apertures}}}\, \underbrace{F.T.\left[f_1(y)\right]}_{\substack{\text{self-interference} \\ \text{(diffraction)}}}$$

Example: Infinite periodic symmetry

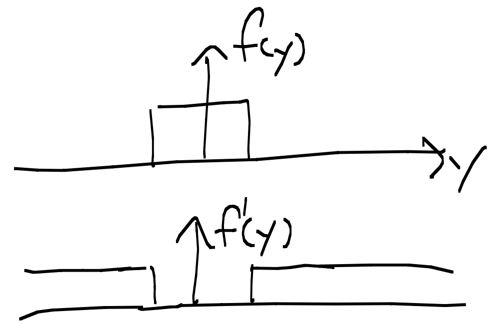


$$F.T.\left[f_\infty(y)\right] = \sum_{n=-\infty}^{\infty} F.T.\left[f_1(y)\right]\,\delta\left(k' - n\frac{2\pi}{d}\right)$$



"diffraction grating"

Complementary apertures



$$f(y) = 1 - f'(y)$$

$$\text{F.T.}[f(y)] = \text{F.T.}[1] - \text{F.T.}[f'(y)]$$

$$= \delta(k) - \text{F.T.}[f'(y)]$$

$$E(x) = \delta(x) - E'(x)$$

For $x \neq 0$

$$E(x) = -E'(x)$$

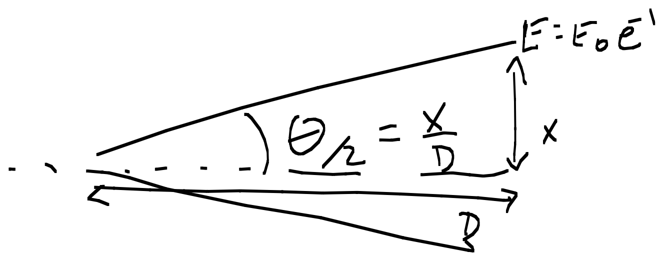
$$I(x) = I'(x)$$ "Babinet's principle"

Gaussian Beam

$$f_g(y) = e^{-\frac{y^2}{w_0^2}} \qquad F.T.\left[f_g(y)\right] = e^{-\frac{k'^2 w_0^2}{4}}$$

Substitute $k' \to \frac{kx}{D}$ so $E(x) = e^{-\frac{k^2 x^2 w_0^2}{4D^2}}$

$E(x)$ falls by $\frac{1}{e}$ when $\frac{k^2x^2w_0^2}{4D^2} = 1$



$$\frac{\theta}{2} = \sqrt{\frac{4}{k^2 w_0^2}} = \frac{2}{k w_0}$$

$$w_0 = \frac{4}{k\theta} = \frac{2\lambda}{\pi\theta}$$

This is the same result as obtained using uncertainty relation in Lecture 1! So behavior of gaussian beam is due to diffraction!