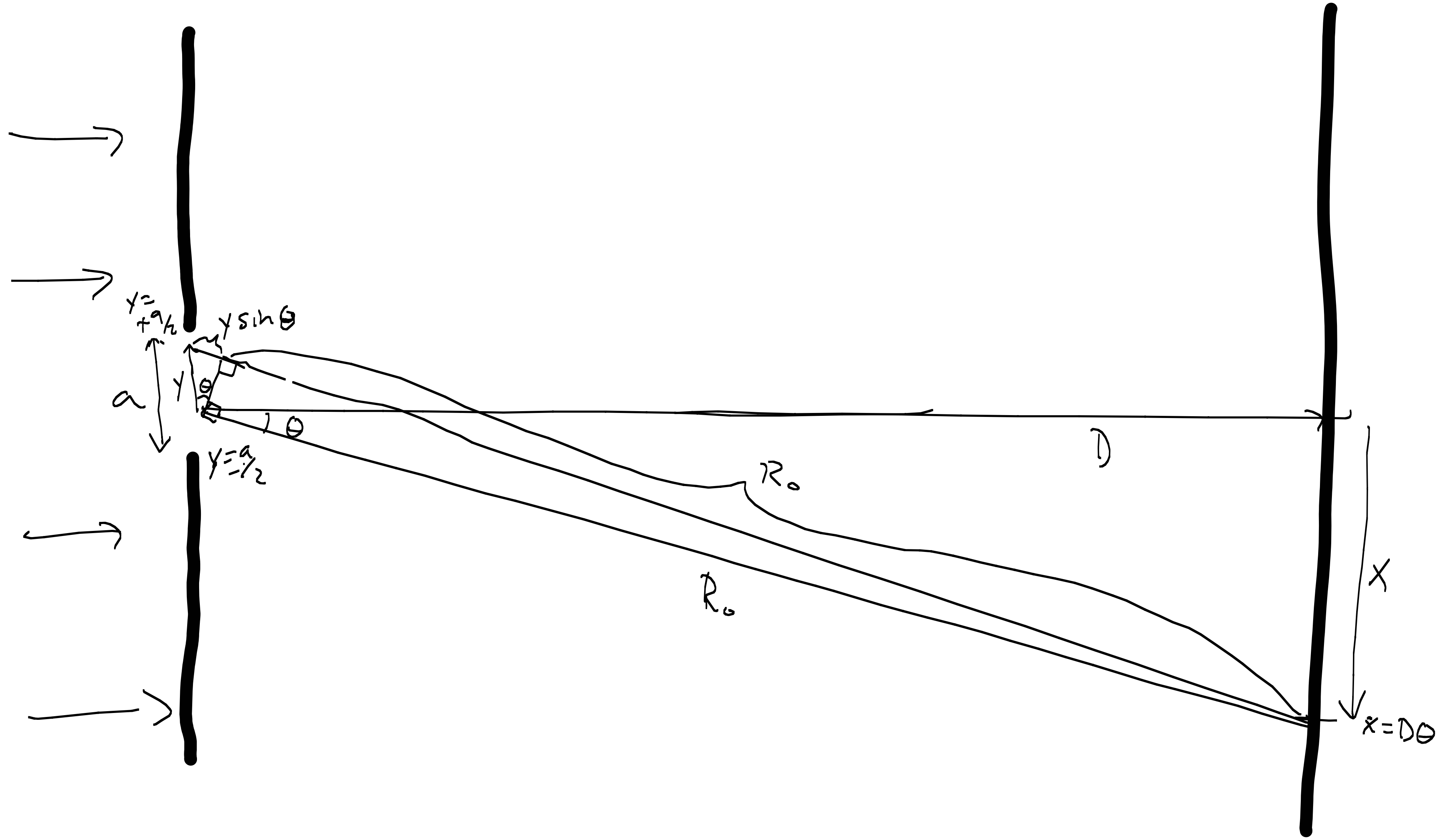
Self-Interference: Diffraction



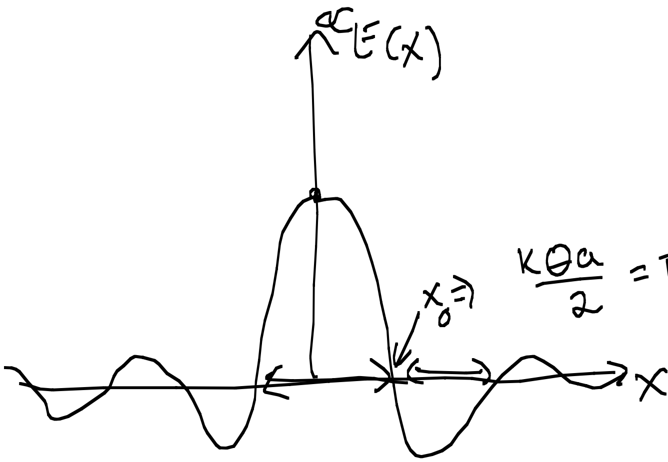
E-field at $x = D\theta$

$$E \propto \int_{-a/2}^{+a/2} e^{ik(R_0 + y\sin\theta)}\,dy = e^{ikR_0}\int_{-a/2}^{+a/2} e^{iky\sin\theta}\,dy$$

$$= e^{ikR_0}\left.\frac{e^{iky\sin\theta}}{ik\sin\theta}\right|_{-a/2}^{+a/2} = e^{ikR_0}\,\frac{e^{ik\sin\theta\frac{a}{2}} - e^{-ik\sin\theta\frac{a}{2}}}{ik\sin\theta}$$

$$= \frac{2e^{ikR_0}\sin\left(k\sin\theta\frac{a}{2}\right)}{k\sin\theta} \underset{x \ll D}{\sim} \frac{\cancel{2}ae^{ikR_0}\sin\frac{k\theta a}{2}}{\cancel{2}\,\frac{k\theta a}{2}} = ae^{ikR_0}\underbrace{\frac{\sin\alpha}{\alpha}}_{\text{"sinc"}}$$

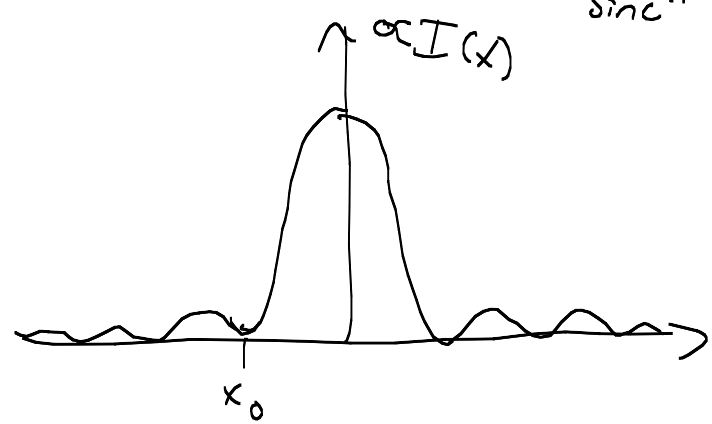
$$\alpha \equiv \frac{k\theta a}{2}$$



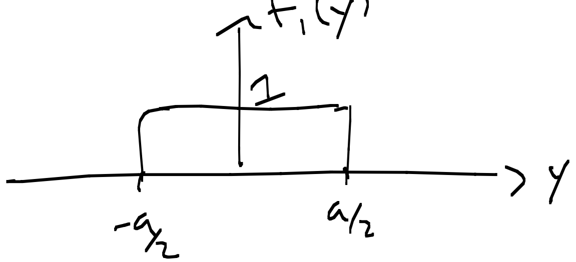
$$\frac{k\theta a}{2} = \pi$$

$$\frac{kx_0 a}{2D} = \pi$$

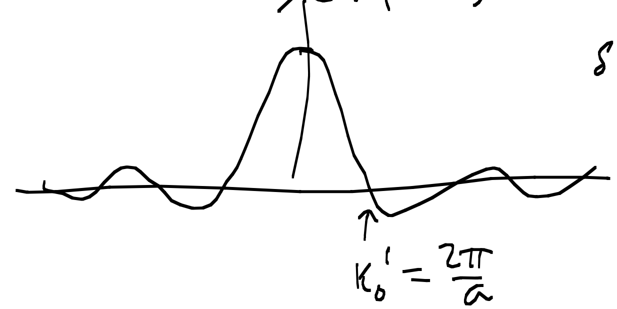
$$x_0 = \frac{2\pi D}{ka} = \frac{\lambda D}{a}$$



Fourier Transform



$\xrightarrow[\text{Transform}]{\text{Fourier}}$



$F_1(k') = \dfrac{\sin \frac{k'a}{2}}{k'a}$

sinc!

To calculate diffraction pattern of an arbitrary aperture,

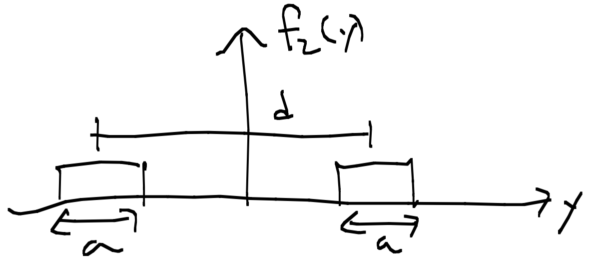
1. find $f(y)$ [aperture transmission fn.]
2. take Fourier Transform of $f(y)$
3. perform variable subs $k' \longrightarrow \dfrac{2\pi x}{D\lambda}$
4. find squared norm to get $I(x)$.

Example: (above)

$$\frac{\sin \frac{k'a}{2}}{k'a} \longrightarrow \frac{\sin \frac{2\pi x}{D\lambda}\frac{a}{2}}{\frac{2\pi x}{D\lambda} a} = \frac{\sin \frac{kxa}{2D}}{\frac{kxa}{D}} = \frac{\sin \frac{ka\theta}{2}}{2k\theta a/2} = \frac{\sin \alpha}{2\alpha}$$

c.f. previous slide

Example: Young's two-slit exp't

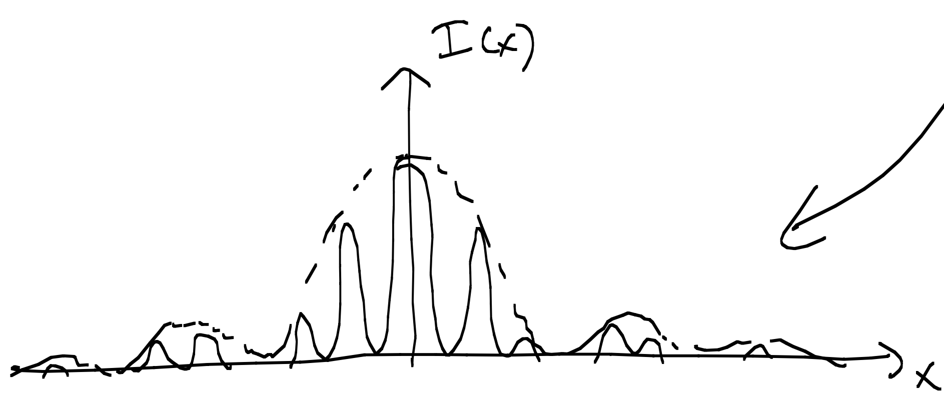


$$F.T.\left[f_2(y)\right] = F.T.\left[f_1\left(y+\frac{d}{2}\right) + f_1\left(y-\frac{d}{2}\right)\right]$$

$$= e^{-ik'\frac{d}{2}} F.T.\left[f_1(y)\right] + e^{+ik'\frac{d}{2}} F.T.\left[f_1(y)\right]$$

$$= F.T.\left[f_1(y)\right]\left(e^{\frac{ik'd}{2}} + e^{-\frac{ik'd}{2}}\right)$$

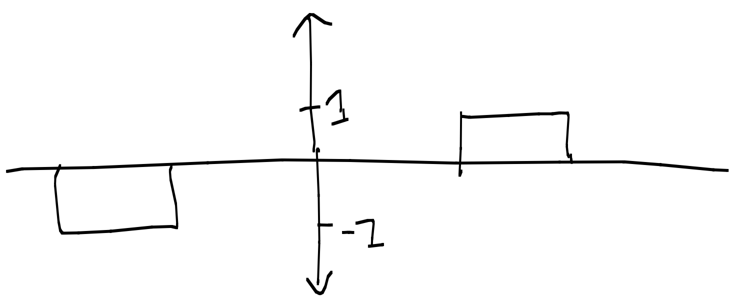
$$= \underbrace{2\cos\frac{k'd}{2}}_{\text{Interference of two apertures}} \underbrace{F.T.\left[f_1(y)\right]}_{\text{self-interference (diffraction)}}$$



$$\left(k' \to \frac{kx}{D}\right) \sim \cos\frac{kxd}{2D}$$

$I(x)$ modulated by $\cos\frac{kxd}{D}$

Example: $\Delta\phi = \pi$

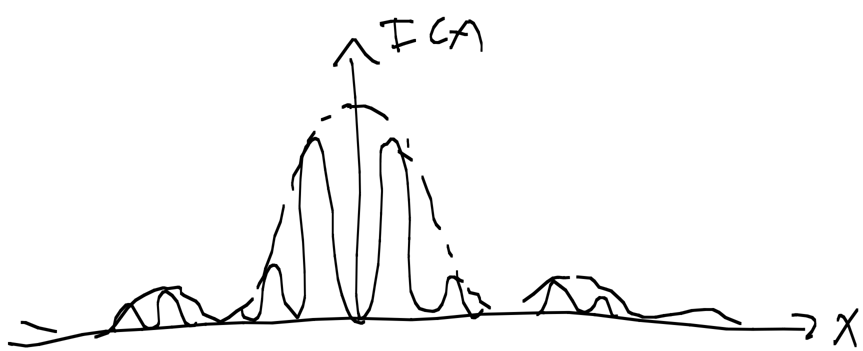


What is the diffracted image?

$$F.T.\left[f_1\left(y-\frac{d}{2}\right) - f_1\left(y+\frac{d}{2}\right)\right]$$

$$= \left(e^{ik'\frac{d}{2}} - e^{-ik'\frac{d}{2}}\right) F.T.\left[f_1(y)\right]$$

$$\left(2i\sin\frac{k'd}{2}\right) F.T.\left[f_1(y)\right]$$

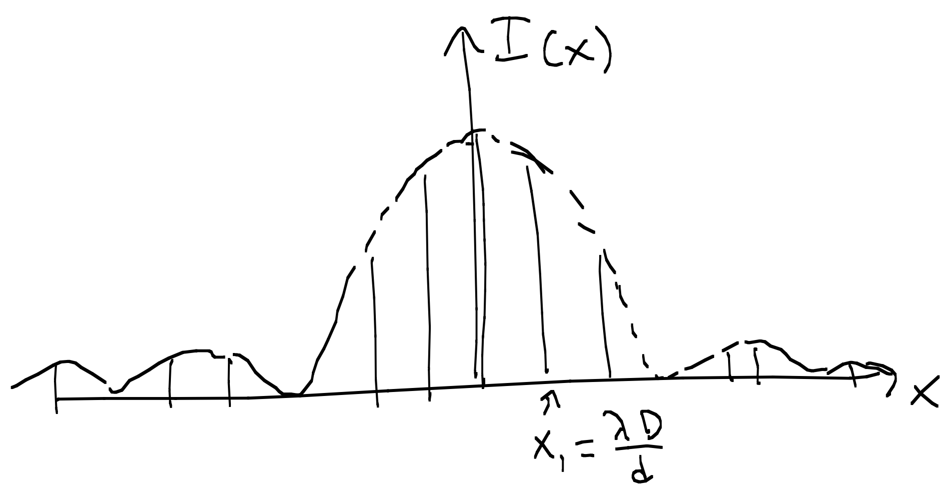


Example: Infinite periodic symmetry

"diffraction grating"

$$F.T.\left[f_\infty(y)\right] = \sum_{n=-\infty}^{\infty} F.T.\left[f_1(y)\right]\,\delta\left(k' - n\frac{2\pi}{d}\right)$$

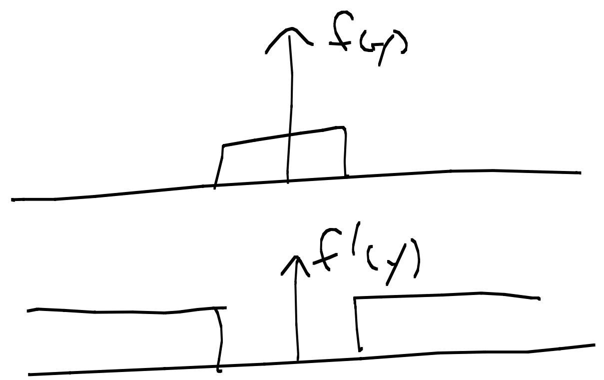
$$\delta\left(\frac{kx}{D} - n\frac{2\pi}{d}\right)$$



$$\frac{kx}{D} = \frac{2\pi}{d}$$

$$x = \frac{2\pi D}{dk} = \frac{\lambda D}{d}$$

complementery apertures



$$f(y) = 1 - f'(y)$$

$$F.T.[f(y)] = F.T.[1] - F.T.[f'(y)]$$

$$= \delta(k') - F.T.[f'(y)]$$

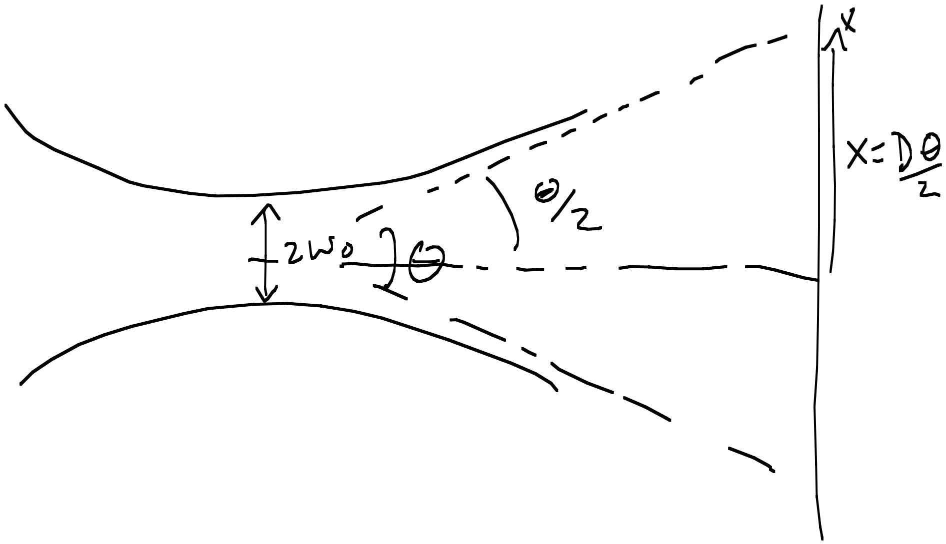
$$E(x) = \delta(x) - E'(x)$$

For $\theta, x \neq 0$,

$$E(x) = -E'(x)$$

$$I(x) = I'(x)$$ "Babinet's principle"

Gaussian Beam



$f_g(y) = e^{-\frac{y^2}{w_0^2}}$ $\quad$ F.T. $[f_g(y)] = e^{-\frac{k'^2 w_0^2}{4}}$

Subst. $k' \to \frac{kx}{D}$ $\quad$ $E(x) = e^{-\frac{k^2 x^2 w_0^2}{4D^2}}$

$E(x)$ falls by $1/e$ when

$$\frac{k^2 x^2 w_0^2}{4D^2} = 1$$

$$x = \sqrt{\frac{4D^2}{k^2 w_0^2}}$$

$$\frac{\cancel{D}\theta}{2} = \frac{2D}{kw_0} = \frac{\lambda \cancel{D}}{\pi w_0}$$

$$w_0 = \frac{2\lambda}{\pi\theta}$$ c.f. lecture 1!

So behavior of gaussian (divergence) is due to diffraction!