UNIVERSITY OF
MARYLAND

Department of Physics
Physics 374 Spring 2009

2nd Midterm Examination, Thursday, May 7, 2009

1.) (10 points) Evaluate the following integral

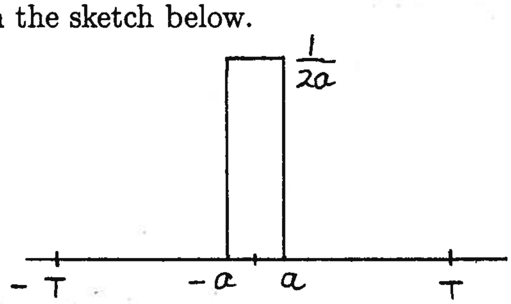
$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dk e^{ikx} \frac{1}{x^2+a^2}, \tag{1}$$

Hint: Consider which integral should be done first.

2.) (15 points) Find the Fourier series for a periodic function obeying $f(t+2T) = f(t)$, where

$$\begin{aligned} f(t) &= 0, \quad && -T < t < -a; \\ f(t) &= \frac{1}{2a} && -a < t < a; \\ f(t) &= 0, && a < t < T. \end{aligned} \tag{2}$$

This function is shown in the sketch below.



3.) (10 points) If $a(t)$ satisfies the differential equation

$$\frac{d^2a(t)}{dt^2} = \delta(t), \tag{3}$$

and the function $x(t)$ is defined by the integral

$$x(t) = \int_{-\infty}^{\infty} d\tau\ a(t-\tau)\ f(\tau), \tag{4}$$

determine the result for $\frac{d^2x(t)}{dt^2}$.

4.) For the function

$$X(\omega) = \frac{1}{-L\omega^2 - i\omega R + \frac{1}{C}} \tag{5}$$

determine the inverse Fourier transform

$$x(t) = \int_{-\infty}^{\infty} d\omega e^{-i\omega t} X(\omega), \tag{6}$$

as follows:

a.) (10 pts.) Where are the poles in the complex plane?

b.) (10 pts.) What are the appropriate contours for closure for each of two possibilities: $t > 0$ and $t < 0$?

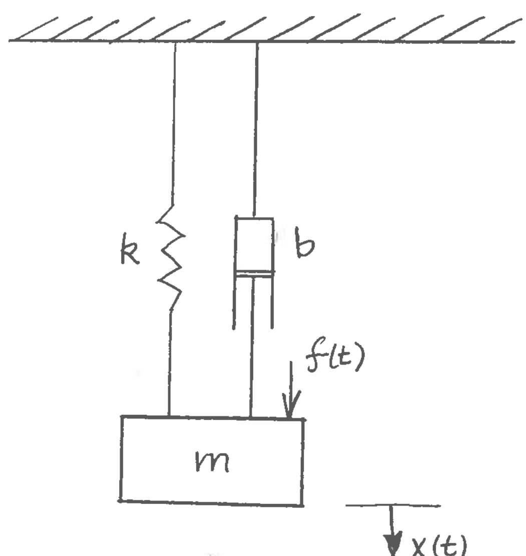
c.) (10 pts.) Determine $x(t)$ for $t < 0$ and give a physical interpretation of the result.

d.) (15 pts.) Determine $x(t)$ for $t > 0$.

5.) For a spring-mass-damper system as in the sketch below, where m is the mass, k is the spring constant and b is the damping constant,

a.) (10 pts.) determine the equation of motion for $x(t)$, the displacement from equilibrium, in response to the applied force $f(t)$.

b.) (10 pts.) Determine the relation between the Fourier transform of $x(t)$ and the Fourier transform of the forcing function $f(t)$.



Midterm Exam. 2 – Solution

1.) using $\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikx} = \delta(x)$ gives

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dk\, e^{ikx} \frac{1}{x^2+a^2} = \int_{-\infty}^{\infty} dx\, \delta(x) \frac{1}{x^2+a^2} = \frac{1}{a^2}$$

2.) Average value $= \frac{a_0}{2} = \frac{1}{2T}\int_{-T}^{T} dt\, f(t) = \frac{1}{2T}\, 2a \frac{1}{2a} = \frac{1}{2T}$

function is even, so all $b_n = 0$

$$a_n = \frac{1}{T}\int_{-T}^{T} dt\, f(t) \cos\left(\frac{n\pi t}{T}\right) = \frac{1}{2aT}\int_{-a}^{a} dt \cos\left(\frac{n\pi t}{T}\right) = \frac{1}{2aT} \left.\frac{\sin\left(\frac{n\pi t}{T}\right)}{\frac{n\pi}{T}}\right|_{-a}^{a}$$

$$= \frac{1}{2an\pi}\left[\sin\left(\frac{n\pi a}{T}\right) - \sin\left(-\frac{n\pi a}{T}\right)\right] = \underline{\underline{\frac{\sin\left(\frac{n\pi a}{T}\right)}{n\pi a}}}$$

$$\text{So } \underline{f(t) = \frac{1}{2T} + \sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi a}{T}\right)}{n\pi a} \cos\left(\frac{n\pi t}{T}\right)} \Rightarrow$$

plot at end shows the result including terms up to

3.)

$$\frac{d^2 x(t)}{dt^2} = \frac{d^2}{dt^2}\int_{-\infty}^{\infty} d\tau\, a(t-\tau) f(\tau) = \int_{-\infty}^{\infty} d\tau \left[\frac{d^2}{dt^2} a(t-\tau)\right] f(\tau)$$

but $\frac{d^2}{dt^2} a(t) = \delta(t)$ – so $\frac{d^2}{dt^2} a(t-\tau) = \delta(t-\tau)$

$$\text{so } \underline{\underline{\frac{d^2 x(t)}{dt^2}}} = \int_{-\infty}^{\infty} d\tau\, \delta(t-\tau) f(\tau) = \underline{\underline{f(t)}}$$

4.) $$\mathcal{X}(\omega) = \frac{-1/L}{\omega^2 + i\omega\frac{R}{L} - \frac{1}{LC}} = \frac{-1/L}{\left(\omega + \frac{iR}{2L}\right)^2 - a^2}\ , \quad a^2 = \frac{1}{LC} - \left(\frac{R}{2L}\right)^2$$

$$= \frac{-1/L}{\left(\omega + \frac{iR}{2L} + a\right)\left(\omega + \frac{iR}{2L} - a\right)}$$

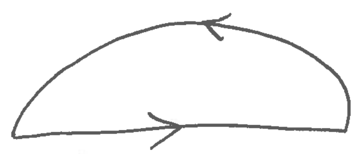
a.) poles at $\omega = -\frac{iR}{2L} - a$ and $\omega = \frac{-iR}{2L} + a$

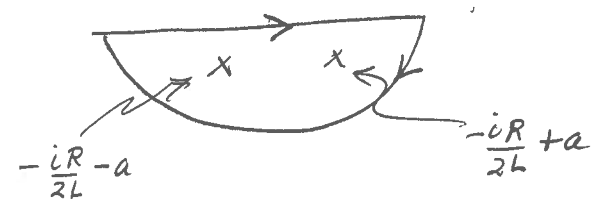
b.) $$x(t) = \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\, \mathcal{X}(\omega)$$

$$e^{-i\omega t} = e^{-iRe(\omega)t}\, e^{Im(\omega)t}$$

for $t > 0$, need $Im(\omega) < 0$
so close in C_-

for $t < 0$, need $Im(\omega) > 0$
so close in C_+





c.) For $t < 0$, there are no poles in C_+.
$\therefore\ x(t) = 0$ by Cauchy's theorem for $t < 0$.

Causality requires this, i.e., a signal out of the RLC circuit must occur after the input is turned on.

d.) For $t > 0$, $$x(t) = -2\pi i\left(-\frac{1}{L}\right)\left[\mathrm{Res}\left(z = -\frac{iR}{2L} - a\right) + \mathrm{Res}\left(z = -\frac{iR}{2L} + a\right)\right]$$

$$= \frac{2\pi i}{L}\left(\left.\frac{e^{-izt}}{z + \frac{iR}{2L} - a}\right|_{z = -\frac{iR}{2L} - a} + \left.\frac{e^{-izt}}{z + \frac{iR}{2L} + a}\right|_{z = \frac{-iR}{2L} + a}\right)$$

4d - continued

$$x(t) = \frac{2\pi i}{L} e^{-R/2L\,t}\left(\frac{e^{iat}}{-2a} + \frac{e^{-iat}}{2a}\right)$$

$$x(t) = \frac{2\pi}{aL} e^{-R/2L\,t} \sin(at) \quad \text{for } t>0$$

5a) Newton's law $\quad f_{spring} = -kx \,,\quad f_{damper} = -b\dot{x}$

$$m\ddot{x} = -kx - b\dot{x} + f(t)$$ where $f(t)$ is applied force

or $$m\ddot{x} + b\dot{x} + kx = f(t) \quad (a.)$$

use $$x(t) = \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\, X(\omega) \,,\quad f(t) = \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\, F(\omega)$$

Eq. (a) becomes

$$\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\left[\left(m(-i\omega)^2 + b(-i\omega) + k\right)X(\omega) - F(\omega)\right] = 0$$

by linear independence of $e^{-i\omega t}$ for different ω, the quantity in square brackets must vanish.

$$\left(-m\omega^2 - i\omega b + k\right)X(\omega) - F(\omega) = 0$$

so $$X(\omega) = \frac{F(\omega)}{-m\omega^2 - i\omega b + k}$$