UNIVERSITY OF MARYLAND

Department of Physics
Physics 374 Spring 2009

Midterm Examination, Thurssday, March 12, 2009

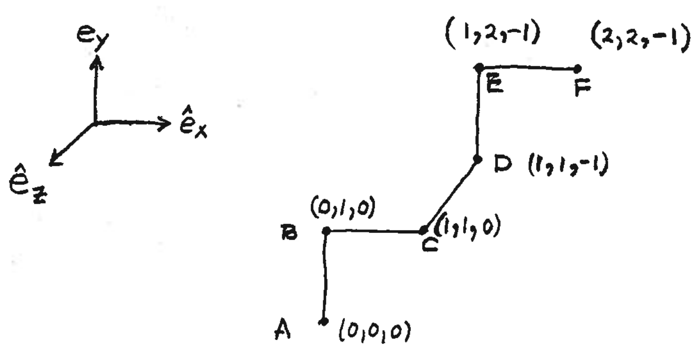
1.) (10 pts.) Calculate the gradient of the following function.

$$f(r,\theta,\phi) = r^3\Big(\frac{3}{2}cos^2\theta - \frac{1}{2}\Big)cos^2\phi \tag{1}$$

2.) (15 pts.) Determine the value of the line integral

$$\int \nabla f(r) \cdot d\mathbf{r} \tag{2}$$

using $\nabla f(r) = x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y + z\hat{\mathbf{e}}_z$ for the path $ABCDEF$ shown below. The coordinates of each point in the sequence $ABCDEF$ are given as (x,y,z) in the figure. *Hint: It may be helpful to figure out what the function $f(r)$ is.*



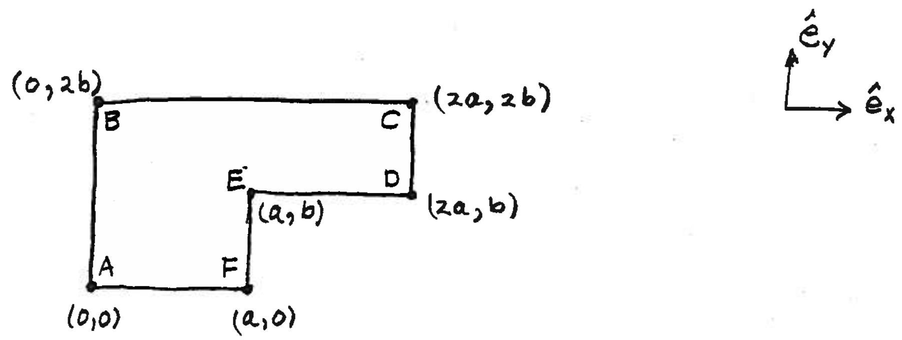
3.) (15 pts.) The function $E = \sqrt{m^2c^4 + p^2c^2}$ is the relativistic total energy of a particle of mass m and momentum $\mathbf{p}$. Often $pc << mc^2$ and in such situations you may expand the energy E about $p = 0$. Determine this expansion with terms up to order p^4 included.

4.) (10 pts.) Determine the result of the expression

$$\nabla \times (\nabla u) \times \mathbf{r} \tag{3}$$

where $u = \mathbf{a} \cdot \mathbf{r}$ with $\mathbf{a}$ being a constant vector.

5.) (15 pts.) Given a vector field $\mathbf{v} = -y\hat{\mathbf{e}}_x + x\hat{\mathbf{e}}_y$, determine the line integral in the $x-y$ plane $\int \mathbf{v} \cdot d\mathbf{r}$ for the closed path $ABCDEFA$ shown below. The coordinates are shown as (x, y) for each point in the sequence $ABCDEF$.



6.) In the following, $\nabla \cdot \mathbf{v} = c$ is constant inside a sphere of radius R that has its center at the origin. Outside the sphere $\nabla \cdot \mathbf{v} = 0$.

a.) (5 pts.) Determine the surface integral $\int \mathbf{v} \cdot \hat{\mathbf{n}} dS$ for the surface of a cube with sides of length $2R$ that is centered on the origin.

b.) (5 pts.) Determine the surface integral $\int \mathbf{v} \cdot \hat{\mathbf{n}} dS$ for the surface of a cube with sides of length R that is centered on the origin.

c.) (10 pts.) What is the vector field (magnitude and direction) at a point that is a distance r from the origin, where $r > R$.

7.) The electric field in an electromagnetic wave is described by $\mathbf{E} = E_0 cos(kz) sin(\omega t)\hat{\mathbf{e}}_y$ where E_0, k and ω are constants.

a.) (5 pts.) Determine $\nabla \cdot \mathbf{E}$ for this wave.

b.) (5 pts.) Determine $\nabla \times \nabla \times \mathbf{E}$.

c.) (5 pts.) The wave equation can be written as $\nabla \times \nabla \times \mathbf{E} + \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$. Use it to determine the relation of wave number k to frequency ω.

Phys 374 Midterm Exam Solution 3/12/09

1.) $\vec{\nabla} f(r,\theta,\phi) = \left(\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial\theta} + \hat{e}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right) r^3\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)\cos^2\phi$

$$= \hat{e}_r\left[3r^2\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)\cos^2\phi\right] + \hat{e}_\theta\left[-3r^2\sin\theta\cos\theta\cos^2\phi\right]$$

$$+\hat{e}_\phi\left[\frac{r^2}{\sin\theta}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)\left(-2\sin\phi\cos\phi\right)\right]$$

2.) $\int_{A\cdots F} \vec{\nabla} f(r)\cdot d\vec{r} = f(r_F) - f(r_A)$ independent of the path.

$\vec{\nabla} f(r) = x\hat{e}_x + y\hat{e}_y + \cancel{z\hat{e}_z} = \vec{r}$ given

$f(r) = \frac{1}{2}r^2$ $\left[\begin{array}{l}\int \vec{\nabla} f\cdot\hat{e}_x dx = \int x\,dx = \frac{1}{2}x^2, \\ \int \vec{\nabla} f\cdot\hat{e}_y dy = \int y\,dy = \frac{1}{2}y^2, \text{ and so on.}\end{array}\right]$

so $\int_{A\cdots F} \vec{\nabla} f\cdot d\vec{r} = \frac{1}{2}\left(r_F^2 - r_A^2\right)$; $r_F^2 = x_F^2 + y_F^2 + z_F^2 = 2^2 + 2^2 + 1^2 = 9$

$r_A^2 = 0$

$$= \underline{\frac{9}{2}}$$

3.) $E = mc^2\sqrt{1+x}$, $x = \frac{p^2c^2}{m^2c^4} = \frac{p^2}{m^2c^2}$

$f(x) = (1+x)^{1/2}$, $f'(x) = \frac{1}{2}(1+x)^{-1/2}$, $f''(x) = -\frac{1}{4}(1+x)^{-3/2}$, ...

$$E = mc^2\left(f(0) + x f'(0) + \frac{x^2}{2} f''(0) + \cdots\right)$$

$$= mc^2\left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots\right)$$

$$\underline{E = mc^2 + \frac{p^2}{2m} - \frac{p^4}{8m^3c^2} + \cdots}$$

4.) $\vec{\nabla} u = \vec{\nabla}(\vec{a}\cdot\vec{r}) = \vec{a}$

so $\vec{\nabla}\times(\vec{\nabla}u)\times\vec{r} = \vec{\nabla}\times\vec{a}\times\vec{r} = \epsilon_{ijk}\,\hat{e}_i \frac{\partial}{\partial x_j} \underbrace{\epsilon_{klm}\, a_l\, x_m}_{(\vec{a}\times\vec{r})_k}$

$$= \underbrace{\epsilon_{kij}\,\epsilon_{klm}}\;\hat{e}_i\; a_l\;\frac{\partial x_m}{\partial x_j}$$

$$= \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)\hat{e}_i\; a_l\;\delta_{mj}$$

$$= \hat{e}_i a_i \underbrace{\delta_{jj}}_{=3} - \hat{e}_j a_j$$

$$= 2\,\hat{e}_i a_i = \underline{\underline{2\vec{a}}}$$

5.) From Stokes theorem

$$\int_C \vec{v}\cdot d\vec{r} = \int_S ds\;\hat{n}\cdot\vec{\nabla}\times\vec{v}$$, $\hat{n} = -\hat{e}_z$ is into page for the path ABCDFA

$$\vec{\nabla}\times\vec{v} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x & 0 \end{vmatrix} = 2\,\hat{e}_z$$

$$\int_C \vec{v}\cdot d\vec{r} = \int_S ds\,(-\hat{e}_z)\cdot(2\hat{e}_z) = -2\int_S ds = -2\times \text{Area}$$

$$= -2\left[(2a)(2b) - a\,b\right] = \underline{\underline{-6ab}}$$

6.) a.) $\int_S \vec{v}\cdot\hat{n}\,ds = \int_V \vec{\nabla}\cdot\vec{v}\,dV$ – the sphere is entirely inside a box with sides of length $2R$, so

$$\int_V \vec{\nabla}\cdot\vec{v}\,dV = \int_{sphere} c\,dV = \underline{\underline{c\cdot\frac{4\pi}{3}R^3}} = c\cdot \text{Volume of sphere}$$

b.) a box with sides of length R is entirely inside sphere

so $\int_V c\,dV = \underline{\underline{c\cdot R^3}}$, where $R^3 =$ volume of box

6.) c.) A point with $r > R$ is outside the sphere

the vector field is spherically symmetric —

so should have the form $v(r)\hat{e}_r$. It

can't have $\hat{e}_\theta$ or $\hat{e}_\phi$ components. For sphere

of radius r,

$$\int_S ds\, \hat{n}\cdot\vec{v} = \int_S ds\, \hat{e}_r \cdot v(r)\hat{e}_r = \int ds\, v(r)$$

$$= v(r)\int ds \qquad (\text{because } r = \text{const.})$$

$$= v(r)\, 4\pi r^2 \qquad (\text{surface area of sphere of radius } r)$$

$$\int_V \vec{\nabla}\cdot\vec{v}\, dV = c\int_{V_{\substack{\text{sphere}\\ R}}} dV = \frac{4\pi}{3}R^3 c \qquad \left(\text{volume of region where } \vec{\nabla}\cdot\vec{v} = c.\right)$$

so $v(r)\, 4\pi r^2 = \frac{4\pi}{3}R^3 c$

or $$\underline{\underline{\vec{v}(r) = \frac{cR^3}{3r^2}\hat{e}_r}}$$

7.) a.) $\vec{\nabla}\cdot\vec{E} = \frac{\partial}{\partial y}\left(E_0 \cos(kz)\sin(\omega t)\right) = \underline{\underline{0}}$

b.) $$\vec{\nabla}\times\vec{E} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & E_y & 0 \end{vmatrix} = -\hat{e}_x \frac{\partial E_y}{\partial z} + \hat{e}_z \underbrace{\frac{\partial E_y}{\partial x}}_{=0} = \underbrace{\hat{e}_x k E_0 \sin(kz)\sin(\omega t)}_{= F\hat{e}_x}$$

$$\vec{\nabla}\times\vec{\nabla}\times\vec{E} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F & 0 & 0 \end{vmatrix} = \hat{e}_y \frac{\partial F}{\partial z} - \hat{e}_z \underbrace{\frac{\partial F}{\partial y}}_{=0} = \underline{\underline{\hat{e}_y k^2 E_0 \cos(kz)\sin\omega t}} = \underline{\underline{k^2\vec{E}}}$$

c.) $\frac{\partial^2\vec{E}}{\partial t^2} = \hat{e}_y E_0 \cos(kz)\frac{\partial^2}{\partial t^2}\sin\omega t = -\omega^2\vec{E}$

so $k^2\vec{E} - \frac{\omega^2}{c^2}\vec{E} = 0$ or $\underline{\underline{\omega = kc}}$