UNIVERSITY OF
MARYLAND

Department of Physics
Physics 374 Spring 2009

Midterm Examination, Thurssday, March 12, 2009

1.) For the function $f(r) = 3r^2 - 2$,

a.) (10 pts.) evaluate $\nabla f(r)$ in Cartesian coordinates.

b.) (10 pts.) Evaluate the line integral $\int \nabla f(r) \cdot d\mathbf{r}$ for the path $ABCDEFGHI$ shown in Figure 1. Coordinates (x, y, z) of the points are shown in Figure 1.

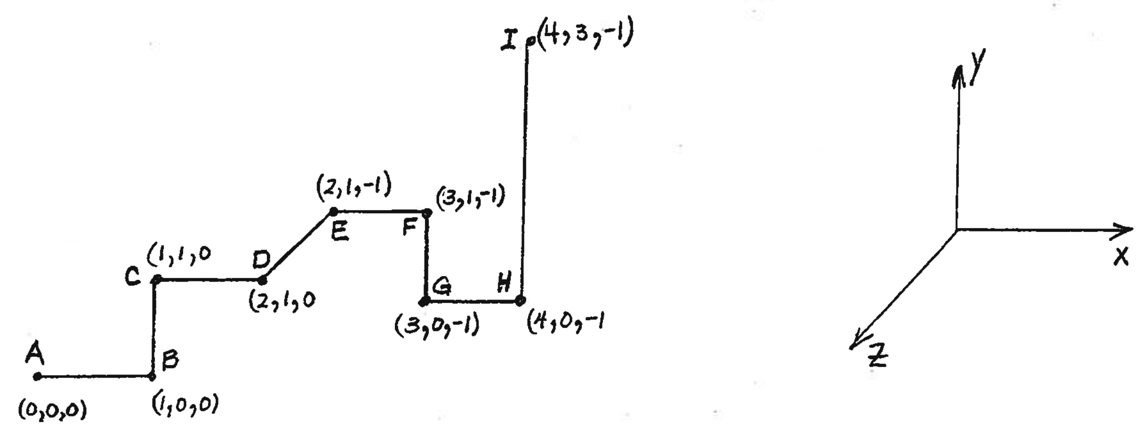


Figure 1

2.) (10 pts.) For a vector field $\mathbf{v} = -2y\hat{e}_x + x\hat{e}_y + (x+y)\hat{e}_z$, evaluate the line integral $\int \mathbf{v} \cdot d\mathbf{r}$ for the *segment DEF* of the path shown in Figure 1. *Note that* $\mathbf{v}$ *is not the gradient of any scalar field.*

3.) (15 pts.) The function $E = \frac{mc^2}{\sqrt{1-v^2/c^2}}$ is the relativistic total energy of a particle of mass m and velocity $\mathbf{v}$. Often $v << c$ and in such situations you may expand the energy E about $v = 0$. Determine this expansion with terms up to order v^4 included.

4.) (15 pts.) The vector field $\mathbf{v}(\mathbf{r})$ obeys $\nabla \times \mathbf{v} = c\hat{e}_y$, where c is a constant.

a.) Evaluate $\int_C \mathbf{v}\cdot d\mathbf{r}$ for the path $ABCDA$ around the inclined plane shown in Figure 2. Coordinates (x, y, z) are shown for each corner of the plane and a view looking toward the origin along the x-direction is shown in Figure 2b so that the 45^o inclination of the plane is clear.

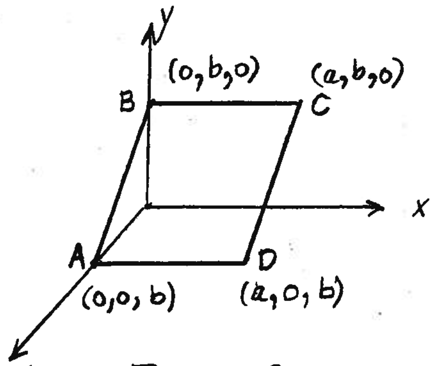


Figure 2a

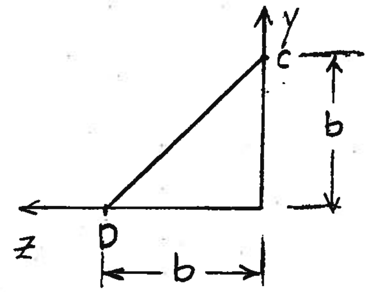


Figure 2b

5.) In the following, $\nabla \cdot \mathbf{v} = c$ is constant inside a cube with dimensions $2R$ on each side and center at the origin. Outside the cube $\nabla \cdot \mathbf{v} = 0$.

a.) (10 pts.) Determine the surface integral $\int \mathbf{v} \cdot \hat{\mathbf{n}} dS$ for the surface of a sphere with radius R and center at the origin.

b.) (10 pts.) Determine the surface integral $\int \mathbf{v} \cdot \hat{\mathbf{n}} dS$ for the surface of a cube with sides of length $3R$ and center at the origin.

6.) (10 pts.) Inside an infinite cylinder of radius R that has its axis along the z-axis, $\nabla \cdot \mathbf{v} = c$, where c is a constant. Outside the cylinder, $\nabla \cdot \mathbf{v} = 0$. Determine the vector field outside the cylinder. *Hint: Consider a finite length L of the cylinder.*

7.) (10 pts.) The kinetic energy of a particle is $\frac{1}{2}mv^2$. Express the kinetic energy in terms of spherical coordinates using $\dot{r}$, $\dot{\theta}$ and $\dot{\phi}$ to denote the rate of change of the spherical coordinates with time. *Possibly useful information:* $\frac{d}{d\theta}\hat{e}_r = \hat{e}_\theta$, $\frac{d}{d\phi}\hat{e}_r = sin\theta\hat{e}_\phi$.

Midterm Exam 3/31/09 Solution

1.a.) $\vec{\nabla}(3r^2-2) = 6r\vec{\nabla}r = 6\vec{r}$ $\quad \left(\vec{\nabla}r^2 = \hat{e}_i \frac{\partial}{\partial x_i} x_j x_j = 2\hat{e}_i x_i = 2\vec{r}\right)$

1b.) $\int_A^I \vec{\nabla}f \cdot d\vec{r} = f(r_I) - f(r_A) = 3r_I^2 - 2 - (3r_A^2 - 2)$

$$r_I^2 = x_I^2 + y_I^2 + z_I^2 = 16 + 9 + 1 = 26 \; , \quad r_A^2 = 0$$

$$\underline{\underline{\int_A^I \vec{\nabla}f \cdot d\vec{r} = 78}}$$

2.) $$\int_D^F \vec{v}\cdot d\vec{r} = \int_D^E \vec{v}\Big|_{\substack{x=2\\y=1}} \cdot (dz\,\hat{e}_z) + \int_E^F \vec{v}\Big|_{\substack{y=1\\z=-1}} \cdot dx\,\hat{e}_x$$

$$= \int_0^{-1} 3\hat{e}_z \cdot dz\,\hat{e}_z + \int_2^3 -2\hat{e}_x \cdot dx\,\hat{e}_x$$

$$= 3z\Big|_0^{-1} - 2x\Big|_2^3$$

$$= -3 - 2(3-2)$$

$$\underline{\underline{\int_D^F \vec{v}\cdot d\vec{r} = \underline{\underline{-5}}}}$$

3.) $E = mc^2(1-x)^{-1/2} \; , \quad x = v^2/c^2$

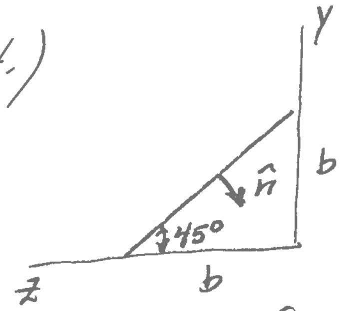
$f(x) = (1-x)^{-1/2} \; , \quad f'(x) = \frac{1}{2}(1-x)^{-3/2} \; , \quad f''(x) = \frac{3}{4}(1-x)^{-5/2}$

$$E = mc^2\left(f(0) + x f'(0) + \frac{x^2}{2} f''(0) + \cdots\right)$$

$$= mc^2\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \cdots\right)$$

$$= mc^2 + \frac{1}{2}mv^2 + \frac{3}{8}\frac{mv^4}{c^2} + \cdots$$

4.)

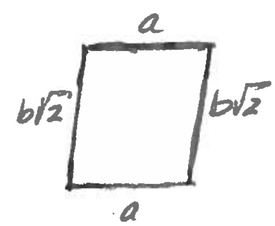


for path ABCDA

$$\hat{n} = -\frac{1}{\sqrt{2}}\hat{e}_z - \frac{1}{\sqrt{2}}\hat{e}_y \quad , \quad \frac{1}{\sqrt{2}} = \cos(45^\circ) = \sin(4$$

$$\text{so} \int_C \vec{v}\cdot d\vec{r} = \int_S \vec{\nabla}\times\vec{v}\cdot\hat{n}\,ds = \int_S c\,\hat{e}_y\cdot\hat{n}\,ds = -\frac{c}{\sqrt{2}}\int ds$$

$$\int ds =$$



$$= ab\sqrt{2}$$

$$\text{so} \int \vec{v}\cdot d\vec{r} = -\frac{c}{\sqrt{2}}\left(ab\sqrt{2}\right) = \underline{\underline{-abc}}$$

5.)

a.) $$\int_{\substack{\text{sphere}\\ R}} \vec{v}\cdot\hat{n}\,ds = \int_{\text{sphere}} \vec{\nabla}\cdot\vec{v}\,dV = c\int_{\text{sphere}} dV = \underline{\underline{\frac{4\pi}{3}R^3 c}}$$

(sphere fits inside cube where $\vec{\nabla}\cdot\vec{v} = c$)

b.) $$\int_{\substack{\text{cube}\\ (3R)\times 3R\times 3R}} \vec{v}\cdot\hat{n}\,ds = \int_V \vec{\nabla}\cdot\vec{v}\,dV$$

here $\vec{\nabla}\cdot\vec{v} = 0$ except inside box with sides $2R$

$$= c\cdot(2R)^3 = \underline{\underline{8R^3 c}}$$

6.) Evaluate $\int_S \vec{v}\cdot\hat{n}\,ds = \int_V \vec{\nabla}\cdot\vec{v}\,dV$ for a cylindrical surface of radius $r > R$ (outside) and height L. By symmetry, $\vec{v}$ must only have a radial component: $\vec{v} = v(r)\,\hat{e}_r$.

$\int_S \vec{v}\cdot\hat{n}\,ds = 2\pi r L\, v(r)$ from sides of cylinder, nothing from top & bottom.

$\int_V \vec{\nabla}\cdot\vec{v}\,dV = \int_{\substack{\text{interior of}\\ r=R}} C\,dV$ because $\vec{\nabla}\cdot\vec{v} = 0$ outside

$= C\cdot\pi R^2 L$ (volume $= \pi R^2 L$)

so $2\pi r L\, v(r) = C\pi R^2 L$

or $\boxed{v(r) = \frac{CR^2}{2r}, \quad \vec{v} = \frac{CR^2}{2r}\hat{e}_r}$

7.) $\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\hat{e}_r) = \dot{r}\hat{e}_r + r\frac{d\hat{e}_r}{d\theta}\dot{\theta} + r\frac{d\hat{e}_r}{d\phi}\dot{\phi}$

$= \dot{r}\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta + r\sin\theta\,\dot{\phi}\,\hat{e}_\phi$

$v^2 = \vec{v}\cdot\vec{v} = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2$

$\frac{1}{2}mv^2 = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right)$