Department of Physics
Physics 374 Spring 2009
Due Thursday, May 7, 2009

1.) An input signal has the Fourier representation

$$v(t) = \int_{-\infty}^{\infty} dt e^{-i\omega t} V(\omega), \tag{1}$$

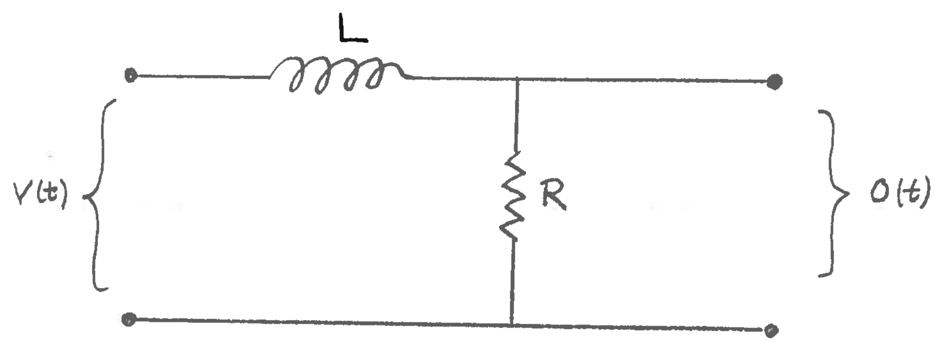
where

$$V(\omega) = \frac{C}{1 + \omega^2/\omega_c^2}. \tag{2}$$

a.) Find the input signal $v(t)$ in the time domain for $t > 0$.

b.) Find the input signal $v(t)$ in the time domain for $t < 0$.

c.) The signal $v(t)$ is passed into a RL circuit as shown below to obtain the output signal $o(t)$.



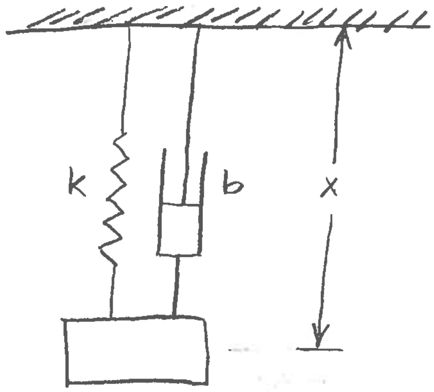
The signal $o(t)$ can also be expressed in terms of a Green's function, $g(t-t')$, which is the response of the circuit when the input voltage is $\delta(t-\tau)$, where τ is a constant. Determine the Green's functions $g(t-\tau)$ for $t > \tau$.

d.) Determine the Green's function for $t < \tau$. Explain your result in terms of causality, i.e., an effect should not occur before its cause.

e.) Use the Green's function to calculate the response as follows,

$$o(t) = \int_{-\infty}^{\infty} d\tau g(t-\tau) v(\tau). \tag{3}$$

2.) This problem concerns the Green's function for a spring-mass-damper system as indicated below.



Let $D(y) = my^2 + by + k$ be a polynomial in the variable y, where m is the mass, b is the damping constant and k is the spring constant.

a.) Show that the equation of motion of the spring-mass-damper system can be written as

$$D\Big(\frac{d}{dt}\Big)x(t) = f(t), \tag{4}$$

where $f(t)$ is an external forcing function.

b.) Show that the Fourier transforms of $x(t)$ and $f(t)$ are related by

$$X(\omega) = G(\omega)\Big[2\pi F(\omega)\Big], \tag{5}$$

where

$$G(\omega) = \frac{\frac{1}{2\pi}}{D(-i\omega)} \tag{6}$$

c.) Using the fact that the inverse Fourier transform of $G(\omega)$ is

$$g(t) = \int_{-\infty}^{\infty} d\omega e^{-i\omega t} G(\omega) \tag{7}$$

, Show that

$$D\Big(\frac{d}{dt}\Big)g(t) = \delta(t). \tag{8}$$

d.) Determine the Green's function $g(t)$ for the spring-mass-damper system.

3.) In three dimensions, the electrostatic potential obeys

$$\nabla^2 V(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\epsilon_0} \tag{9}$$

If the photon had a small mass μ, the equation would be modified to the form

$$(\nabla^2 - \mu^2)V(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\epsilon_0} \tag{10}$$

In this problem assume that the photon has a mass.

Three-dimensional Fourier transforms can be written as

$$V(\mathbf{r}) = \frac{1}{(2\pi)^3}\int d^3k e^{-i\mathbf{k}\cdot\mathbf{r}}\tilde{V}(\mathbf{k})$$
$$\tilde{V}(\mathbf{k}) = \int d^3r e^{i\mathbf{k}\cdot\mathbf{r}}V(\mathbf{r}) \tag{11}$$

. Note: the factors of 2π and the sign of $\mathbf{k}$ have been chosen differently from Snieder's choices.

a.) Determine the Fourier transform of a 3D δ-function, $\delta^{(3)}(\mathbf{r})$.

b.) Show that the Fourier transforms of $V(\mathbf{r})$ and $\rho(\mathbf{r})$ are related by

$$\tilde{V}(\mathbf{k}) = \frac{-1}{\mathbf{k}^2 + \mu^2}\left[-\frac{\rho(\mathbf{k})}{\epsilon_0}\right] \tag{12}$$

c.) Find the Green's function $G(\mathbf{r})$, whose Fourier transform, $\tilde{G}(\mathbf{k})$, is the first factor in the equation above. First show that the angle integrals $\int d\phi \int d\theta sin\theta$ involved in integration over d^3k can be done by choosing $\mathbf{r}$ to lie along the z-axis so that $\mathbf{k}\cdot\mathbf{r} = krcos\theta$. Then use $y = cos\theta$ instead of θ as the integration variable. The result of performing the angle integrals should be

$$G(r) = \frac{1}{(2\pi)^2}\int_0^\infty dk k^2 \frac{e^{ikr} - e^{-ikr}}{ikr}\tilde{G}(\mathbf{k}). \tag{13}$$

Next show that the integral of the part involving e^{-ikr} from 0 to ∞ can be transformed into an integral over k involving e^{ikr} from $-\infty$ to 0. After those steps, the integration can be done using the residue theorem.