Homework 1

1.

a) pressure $p \sim [F/A] = [Ma/A] = \left[\frac{M}{LT^2}\right]$

b) shear sress $\tau = \mu \frac{\partial v}{\partial z}$ where v :velocity, z :distance. With $\tau \sim [p]$, we have $\mu \sim \left[\frac{M}{LT}\right]$

c) $\left[\Phi \frac{dp}{dx}^{\alpha} \mu^{\beta} R^{\gamma}\right] = [1]$, $\left(\frac{L^3}{T}\right)\left(\frac{M}{L^2T^2}\right)^{\alpha}\left(\frac{M}{LT}\right)^{\beta} L^{\gamma} = 1$

$L : 3 - 2\alpha - \beta + \gamma = 0$

$M : \alpha + \beta = 0$

$T : -1 - 2\alpha - \beta = 0$

We get $\alpha = -1, \beta = 1, \gamma = -4$.Therefore, $\Phi = const \frac{\partial p/\partial x}{\mu} R^4$

2.

$R \sim [L], F \sim \left[\frac{ML}{T^2}\right], \mu \sim \left[\frac{M}{LT}\right], v \sim \left[\frac{L}{T}\right]$

$\left[FR^{\alpha}\mu^{\beta}v^{\gamma}\right] = [1], \left(\frac{ML}{T^2}\right)(L)^{\alpha}\left(\frac{M}{LT}\right)^{\beta}\left(\frac{L}{T}\right)^{\gamma} = 1$

$M : 1 + \beta = 0$

$L : 1 + \alpha - \beta + \gamma = 0$

$T : -2 - \beta - \gamma = 0$

We get $\alpha = \beta = \gamma = -1$. Therefore, $F = const.R\mu v$

3.

$\vec{A}, \vec{B}$ can be expressed by colume matrices A, B in one coordinate and A', B' in the other coordinate. They are related by orthogonal transformation R with $R^T R = 1$ (consider real number case). Then $\vec{A}' \cdot \vec{B}' = A'^T B' = (RA)^T (RB) = A^T R^T R B = A^T B = \vec{A} \cdot \vec{B}$ is invariant under orthogonal transformation. (In component form $A'_i B'_i = R_{ij} A_j R_{ij'} B_{j'} = \delta_{jj'} A_j B_{j'} = A_j B_j$.)

4.

Let $\vec{x} = \alpha\vec{a} + \beta\vec{b} + \gamma\vec{c}$. Construct $\vec{b} \times \vec{c}$ and multiply both sides. We get $\vec{x} \cdot \left(\vec{b} \times \vec{c}\right) = \alpha\vec{a} \cdot \left(\vec{b} \times \vec{c}\right) + 0$. Then $\alpha = \frac{\vec{x} \cdot \left(\vec{b} \times \vec{c}\right)}{\vec{a} \cdot \left(\vec{b} \times \vec{c}\right)}$. Likewise, $\beta = \frac{\vec{x} \cdot (\vec{c} \times \vec{a})}{\vec{b} \cdot (\vec{c} \times \vec{a})}$,$\gamma = \frac{\vec{x} \cdot \left(\vec{a} \times \vec{b}\right)}{\vec{c} \cdot \left(\vec{a} \times \vec{b}\right)}$.

5.

a) Let LHS be $\varepsilon_{ijk}\varepsilon_{ilm}$ and RHS be $\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$.

If $j = k$ or $l = m$, LHS=$\varepsilon_{ijj}\varepsilon_{ilm} = 0$, RHS=$\delta_{jl}\delta_{jm} - \delta_{jm}\delta_{jl} = 0$.

Consider $j \neq k$ and $l \neq m$.

If $j = l$ and $k = m$, LHS=$\varepsilon_{ijk}\varepsilon_{ijk} = 1$, RHS=$\delta_{jj}\delta_{kk} - \delta_{jk}\delta_{kj} = 1 - 0 = 1$.

If $j = l$ and $k \neq m$, LHS=$\varepsilon_{ijk}\varepsilon_{ijm} = 0$, RHS=$\delta_{jj}\delta_{km} - \delta_{jm}\delta_{kj} = 0 - 0 = 0$.

If $j \neq l$ and $k = m$, LHS=$\varepsilon_{ijk}\varepsilon_{ilk} = 0$, RHS=$\delta_{jl}\delta_{kk} - \delta_{jk}\delta_{kl} = 0 - 0 = 0$.

If $j \neq l$ and $k \neq m$, LHS=$\varepsilon_{ijk}\varepsilon_{ilm} = 0$, RHS=$\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl} = 0 - 0 = 0$.

Similarly for $j = m$ or $j \neq m$ and $k = l$ or $k \neq l$, LHS=RHS.

Therefore we prove $\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$.

b) (Repeated indeces means sum over that indeces) $\left(\vec{A}\times\vec{B}\right)\cdot\left(\vec{C}\times\vec{D}\right) = \varepsilon_{ijk}A_jB_k\varepsilon_{ilm}C_lD_m = \left(\delta_{jl}\delta_{km}-\delta_{jm}\delta_{kl}\right)A_jB_kC_lD_m = A_jC_jB_kD_k - A_jD_jB_kC_k = \left(\vec{A}\cdot\vec{C}\right)\left(\vec{B}\cdot\vec{D}\right) - \left(\vec{A}\cdot\vec{D}\right)\left(\vec{B}\cdot\vec{C}\right)$

6.

a) $\det\left[\vec{A}\vec{B}\vec{C}\right] = \det\begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix} = A_1B_2C_3 + A_2B_3C_1 + A_3B_1C_2 - A_1B_3C_2 - A_2B_1C_3 - A_3B_2C_1 = \varepsilon_{ijk}A_iB_jC_k$

b) In n dimension,

define $\varepsilon_{ijkl...} = \left\{\begin{array}{c} +1\,,\ \text{if}\ \{ijkl...\}\ \text{is an even permutation of}\ \{1,2,3..,n\} \\ -1,\ \text{if}\ \{ijkl...\}\ \text{is an odd permutation of}\ \{1,2,3..,n\} \\ 0,\ \text{otherwise} \end{array}\right\}.$

Then $\det\left[\vec{A}\vec{B}\vec{C}\vec{D}...\right] = \varepsilon_{ijkl...}A_iB_jC_kD_l...$