Midterm2

1.

In rectangular coordinate,

$\nabla \frac{a}{r} = a\hat{e}_i \partial_i \frac{1}{\sqrt{x^2+y^2+z^2}} = -a\hat{e}_i \frac{x_i}{\left(\sqrt{x^2+y^2+z^2}\right)^3} = -\frac{a}{r^2}\hat{r}.$

In spherical coordinate,

$\nabla \frac{a}{r} = a\hat{r}\partial_r \frac{1}{r} + \frac{1}{r}\hat{\phi}\partial_\phi 0 + \frac{1}{r\sin\theta}\hat{\theta}\partial_\theta 0 = -\frac{a}{r^2}\hat{r}.$

2.

$\int \vec{V} \cdot d\vec{s} = \int (R\cos\theta)\,\hat{z} \cdot \hat{r} R^2 d\cos\theta d\phi = 2\pi R^3 \int_{-1}^{1} (\cos\theta)^2 \, d\cos\theta = \frac{4\pi R^3}{3}.$

$\int \nabla \cdot \vec{V} d\tau = \int \partial_z z d\tau = \int d\tau = \frac{4\pi R^3}{3}.$

3.

$\nabla \times \left(\frac{1}{2}\vec{B} \times \vec{r}\right) = \frac{1}{2}\hat{e}_i \varepsilon_{ijk} \partial_j \left(\vec{B} \times \vec{r}\right)_k = \frac{1}{2}\hat{e}_i \varepsilon_{ijk} \partial_j \varepsilon_{klm} B_l r_m = \frac{1}{2}\hat{e}_i \varepsilon_{ijk} \varepsilon_{klm} \partial_j B_l r_m$

$= \frac{1}{2}\hat{e}_i (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \partial_j B_l r_m = \frac{1}{2}\hat{e}_i B_i \partial_j r_j - \frac{1}{2}\hat{e}_i B_j \partial_j r_i = \frac{1}{2}\vec{B}\nabla \cdot \vec{r} - \frac{1}{2}\vec{B} = \vec{B}.$

4.

$f(x) = \frac{1}{\sqrt{2\pi}} \int dk e^{ikx} f(k).$

$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int dx \frac{1}{2\pi} \int dk_1 e^{-ik_1 x} f(k_1)^* \int dk_2 e^{ik_2 x} f(k_2)$

$= \int dk_1 f(k_1)^* \int dk_2 f(k_2) \frac{1}{2\pi} \int dx e^{-i(k_1-k_2)x}$

$= \int dk_1 f(k_1)^* \int dk_2 f(k_2)\, \delta(k_1 - k_2) = \int dk\, |f(k)|^2.$

5.

If $x > 0$, close the contour in the upper plane for $\lim_{p\to i\infty} e^{ipx} = 0$.The pole is at $p = i\varepsilon$ in the upper plane.

$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{i}{p-i\varepsilon} e^{ipx} dp = \frac{1}{2\pi} 2\pi i \left(i e^{i(i\varepsilon)x}\right) = -e^{-\varepsilon x}.$

If $x < 0$, close the contour in the lower plane and no pole there. So the integral is zero.